ECE 146B Lab: Digital beamforming in a mmWave MIMO uplink

Lab Objectives: The goal of this lab is to explore interference modeling and suppression for a mmWave MIMO uplink in which the base station is capable of performing digital beamforming.

1 Millimeter wave background

Technically, millimeter wave (mmWave) refers to frequency bands from 30 to 300 GHz (wavelengths of 10 mm to 1 mm). Unlike the relatively crowded spectrum in today's cellular and WiFi bands, there are vast amounts of spectrum in these bands. Advances in radio frequency (RF) integrated circuits (ICs) are making these bands accessible for commercial mass market applications. Emerging 5G cellular systems aim to utilize spectrum at 28 GHz. There are very large amounts of unlicensed spectrum in the 60 GHz band (it used to be 7 GHz, but it has been recently increased to 14 GHz). Part of the 60 GHz band suffers larger propagation loss due to oxygen absorption: about 16 dB/km on top of standard propagation loss. Of course, over shorter ranges, such as 100m, this is not a dealbreaker. There is also increasing interest in spectrum above 100 GHz.

There are two key features of mmWave worth noting:
• The tiny carrier wavelengths enable the realization of antenna arrays with a large number of elements in compact form factors (e.g., comparable to those of WiFi access points and handhelds), which allows us to form highly directive, electronically steerable, beams.
• The bandwidth produced by RF circuits tuned to a given carrier frequency f_0 is proportional (e.g., up to 10% of f_0), and therefore scales up as we increase the carrier frequency. Thus, at mmWave frequencies, we are typically considering bandwidths of the order of GHz.

In this lab, we consider a picocellular base station with a large number of antennas, supporting a large number of simultaneous users. This is called *space-division multiple access (SDMA)* or *multiuser MIMO (MU-MIMO),* or more dramatically, *massive MIMO.* We assume that the base station employs a DSP-centric architecture as shown in Figure 1.

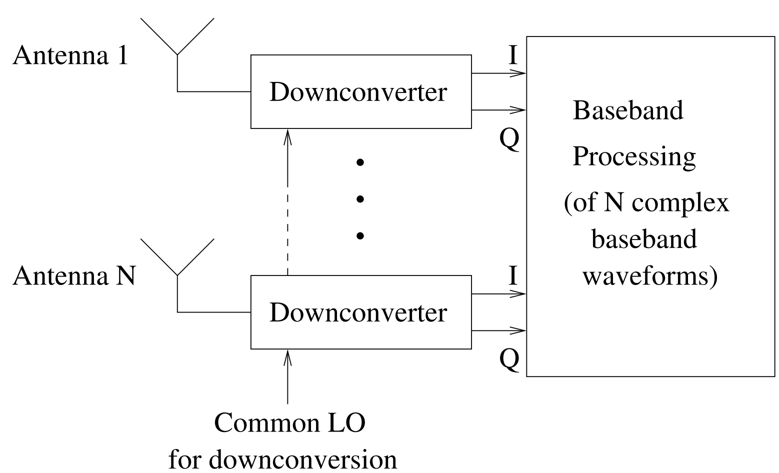

Figure 1: MIMO signal processing architecture. There is one "RF chain" per antenna, downconverting the signal received at that antenna to I and Q components.

Signal processing architecture: For an N antenna receiver, we downconvert the RF signals at the outputs of the antenna elements (using the same LO frequency and phase, and filters with identical responses, in each such "RF chain"). The relationship between the complex envelopes corresponding to the different antenna elements is discussed in the next section. Once the I and Q components for these complex envelopes are obtained, they would typically be sampled and quantized using analog-to-digital converters (ADCs), and then processed digitally. Such a DSP-centric signal processing architecture, depicted in Figure 1, allows the implementation of sophisticated MIMO algorithms in today's cellular and WiFi systems. While the figure depicts a receiver architecture, an entirely analogous block diagram can be drawn for a MIMO transmitter, simply by reversing the arrows and replacing downconverters by upconverters.

While the DSP-centric architecture depicted in Figure 1 has been key to enabling the widespread deployment of low-cost MIMO transceivers, it is challenging at mmWave. Scaling RF design for a large number of elements on both receive and transmit is difficult, as is analog-to-digital conversion at high symbol rates. In this lab, however, we ignore these issues, assuming that these bottlenecks will eventually be overcome. We are motivated by recent research showing that, while such architectures are challenging, there are certain regimes in which they are particularly attractive and on the cusp of feasibility: for example, as we scale up the number of antennas, if the system load (ratio of number of simultaneous users to number of antennas) is small, then we can get away with lower precision ADC and more relaxed RF design specifications.

2 The linear array

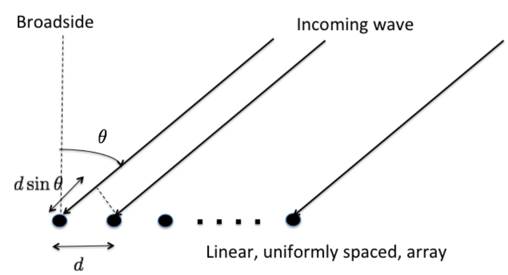

Figure 2: A plane wave impinging on a linear array.

Consider a plane wave impinging on the uniformly spaced linear array shown in Figure 2. We see that the wave sees slightly different path lengths, and hence different phase shifts, in reaching different antenna elements. The path length difference between two successive elements is given by $\ell = d \sin\theta$, where d is the inter-element spacing, and θ the angle of arrival (AoA) relative to the broadside. The corresponding phase shift across successive elements is given by $\Omega = 2\pi\ell/\lambda = 2\pi d \sin\theta/\lambda$, where λ denotes the wavelength. Another way to get the same result: the delay difference between successive elements is $\tau = \ell/c$, where c is the speed of wave propagation (equal to 3×10^8 m/s in free space). For carrier frequency f_c, the corresponding phase shift is $\Omega = 2\pi f_c \tau = 2\pi f_c d \sin\theta/c$. The two expressions are equivalent, since $\lambda = \frac{c}{f_c}$.

According to our convention, the situation depicted in Figure 2 corresponds to positive θ (clockwise from broadside). Thus, θ takes values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, although we often restrict the field of view of the array to a smaller range (more on this later). Numbering antenna elements from left to right in Figure 2, we see that, for positive θ, the first element sees the longest path length. Let us see what effect it has on the received waveform, that we model as usual in complex baseband.

The narrowband assumption: What is the effect of the differences in delays seen by successive elements? Suppose that the wave impinging on element 1 is represented as

$$u_p(t) = u_c(t) \cos 2\pi f_c t - u_s(t) \sin 2\pi f_c t = \mathrm{Re}\left(u(t) e^{j2\pi f_c t}\right)$$

where $u(t) = u_c(t) + j u_s(t)$ is the complex envelope, assumed to be of bandwidth W. Suppose that the bandwidth $W \ll f_c$: this is the so-called "narrowband assumption," which typically holds in most practical settings. For the scenario shown in the figure, the wave arrives $\tau = \ell/c$ time units earlier at element 2. The wave impinging on element 2 can therefore be represented as

$$v_p(t) = u_c(t+\tau) \cos 2\pi f_c (t+\tau) - u_s(t+\tau) \sin 2\pi f_c (t+\tau) = \mathrm{Re}\left(u(t+\tau) e^{j\Omega} e^{j2\pi f_c t}\right)$$

where $\Omega = 2\pi f_c \tau$. Thus, the complex envelope of the wave at element 2 is $v(t) = u(t + \tau)e^{j\Omega}$. The time shift τ has two effects on the complex envelope: a time shift in the baseband waveform u, along with a phase rotation Ω due to the carrier. However, for most settings of interest, the time shift in the baseband waveform can be ignored. To see why, suppose that the array parameters are such that Ω is of the order of 2π or less, in which case τ is of the order of $\frac{1}{f_c}$ or less. Under the narrowband assumption, the time shift τ produces little distortion in u. To see this, note that

$$u(t + \tau) \leftrightarrow U(f)e^{j2\pi f\tau}$$

As f varies over a range W, the frequency-dependent phase change produced by the time shift varies over a range $2\pi W\tau \sim 2\pi W/f_c \ll 2\pi$ for $W \ll f_c$. Thus, we can ignore the effect of the time shift on the complex envelope, and model the complex envelope at element 2 as $v(t) \approx u(t)e^{j\Omega}$. Similarly, for element 3, the complex envelope is well approximated as $u(t)e^{j2\Omega}$.

Array response and spatial frequency: Under the narrowband assumption, if the complex envelope at element 1 is $u(t)$, then the complex envelopes at the various elements can be collected into a vector $u(t)\mathbf{a}$, where

$$\mathbf{a} = (1, e^{j\Omega}, e^{j2\Omega}, ..., e^{j(N-1)\Omega})^T \tag{1}$$

is the *array response* for a particular AoA. The linear progression in phase across antenna elements (i.e, across space) is analogous to the linear progression of phase in time, $e^{j2\pi f_0 t}$, for a complex exponential at frequency f_0. Thus, we call

$$\Omega = \Omega(\theta) = 2\pi d \sin\theta/\lambda \tag{2}$$

the *spatial frequency* corresponding to AoA θ.

We denote the array response (1) by $\mathbf{a}(\theta)$, or $\mathbf{a}(\Omega)$, where we economize on notation by using $\mathbf{a}(\cdot)$ to indicate dependence on either θ or Ω as convenient. We will see that it is more natural to work with spatial frequency when characterizing how the spatial responses for different mobiles interact at the base station.

Reciprocity: While Figure 2 depicts an antenna array receiving a wave, exactly the same reasoning applies to an antenna array emitting a wave. In particular, the principle of reciprocity tells us that the propagation channel from transmitter to receiver is the same as that from receiver to transmitter. Thus, the array response of a linear array for angle of arrival θ is the same as the array response for angle of departure θ.

3 A Picocellular Uplink Model

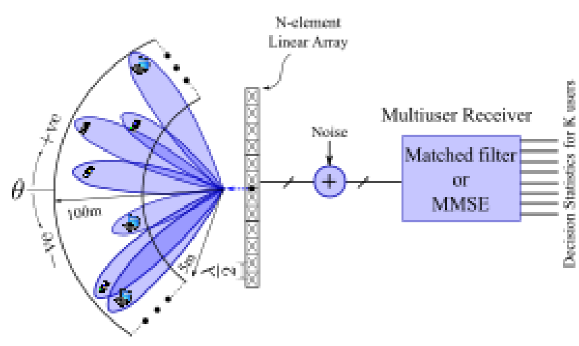

Figure 3: A mmWave uplink picocell

Figure 3 depicts the system model. The base station has a linear array with N antenna elements, with inter-element spacing of $\frac{\lambda}{2}$. The sector width is 120 degrees: that is, the angles of arrival from the mobiles lie in $[-\frac{\pi}{3}, \frac{\pi}{3}]$. There are K mobiles that transmit simultaneously, each with a line of sight (LoS) channel to the base station. Assuming Nyquist signaling, and making the unrealistic assumption that the symbols for all mobiles line up at the base station, we can obtain a simple but useful discrete-time symbol rate model at the base station. The $N \times 1$ spatial signature of user k ($1 \leq k \leq K$) on the antenna array is given by

$$\mathbf{u}_k = A_k e^{j\phi_k} \mathbf{a}(\Omega_k) \tag{3}$$

where A_k is the received amplitude; ϕ_k is the phase, assumed to be independent across users and uniform over $[0, 2\pi]$, and Ω_k is the spatial frequency. The $N \times 1$ (corresponding to the N antenna elements) complex baseband received signal vector corresponding to the nth symbol is now given by

$$\mathbf{y}[n] = \sum_{k=1}^{K} b_k[n]\mathbf{u}_k + \mathbf{w}[n] \tag{4}$$

where $b_k[n]$ is the nth symbol transmitted by user k.

The noise, denoted by $\mathbf{w}[n]$, is $CN(\mathbf{0}, 2\sigma^2\mathbf{I})$ (i.e., the real and imaginary components for each antenna element are i.i.d. $N(0, \sigma^2)$ random variables). Note that the symbol sequences for different users are independent. We also assume them to be zero mean and normalized (without loss of generality) to unit average magnitude: $E[|b_k[n]|^2] = 1$ (symbol scaling can be absorbed into the amplitude parameters A_k in (3)).

From the point of view of a given user, say user j, the other users are interference, and the model looks like

$$\mathbf{r}[n] = b_j[n]\mathbf{u}_j + \mathbf{I}_j[n] \tag{5}$$

where $\mathbf{u}_j$ is the "desired" signal vector and

$$\mathbf{I}_j[n] = \sum_{k \neq j} b_k[n]\mathbf{u}_k + \mathbf{w}[n] \tag{6}$$

is the contribution of interference plus noise to the received vector (from the point of view of user j).

4 Linear receivers

We assume that the base station applies a different linear correlator for each user to the received vector to produce its decision statistic:

$$Z_j[n] = \mathbf{c}_j^H \mathbf{r}[n] \tag{7}$$

From (4), the correlator output can be written as a sum of desired and undesired terms:

$$Z_j[n] = b_j[n]\mathbf{c}_j^H\mathbf{u}_j + \left[\sum_{k \neq j} b_k[n]\mathbf{c}_j^H\mathbf{u}_k + \mathbf{c}_j^H\mathbf{w}[n]\right] \tag{8}$$

where the term in square brackets contains the undesired interference plus noise contributions to the correlator output. Using (5) and (6), we may write this more compactly as

$$Z_j[n] = b_j[n]\mathbf{c}_j^H\mathbf{u}_j + \mathbf{c}_j^H\mathbf{I}_j[n] \tag{9}$$

For any such correlator, the output signal-to-interference-plus-noise ratio (SINR) is defined as the ratio of the powers of the desired and undesired terms:

$$SINR_j = \frac{E[|b_j[n]\mathbf{c}_j^H\mathbf{u}_j|^2]}{E[|\mathbf{c}_j^H\mathbf{I}_j[n]|^2]} \tag{10}$$

For the model (4), assuming that the symbols are uncorrelated across users (and normalized to unit energy), we obtain that

$$SINR_j = \frac{|\mathbf{c}_j^H\mathbf{u}_j|^2}{\sum_{k=1,k\neq j}^{K} |\mathbf{c}_j^H\mathbf{u}_k|^2 \ + 2\sigma^2||\mathbf{c}_j||^2} \tag{11}$$

We will consider two different linear correlators. The first is a spatial matched filter, which maximizes the desired signal contribution without accounting for the interference. It is optimal for WGN, since it maximizes the SNR, but incurs substantial degradation in performance in the presence of interference. The second is the linear Minimum Mean Squared Error (MMSE) receiver, which maximizes the SINR.

4.1 Spatial matched filter

If there were no interference, then we can maximize the received power for user j by employing a *spatial matched filter* or *spatial correlator.* This amounts to setting $\mathbf{c}_j = \mathbf{u}_j$ for user j. Setting the interference terms to zero, we see that the SNR attained by the spatial matched filter is given by

$$SNR_j = \frac{||\mathbf{u}_j||^2}{2\sigma^2} = \frac{A_j^2 N}{2\sigma^2} \tag{12}$$

where we have plugged in (3) to obtain the second equality (with our definition of array response, $\mathbf{a}_j^H \mathbf{a}_j = N$.) That is, spatial matched filtering leads to an SNR enhancement of N, the so-called *receive beamforming gain:* the signal amplitudes add up coherently across the N antennas, so that signal power is enhanced by N^2, while the noise variances add up across the antennas, so that noise power scales by N.

When we do have interference, the SINR expression (11) for the matched filter receiver is given by

$$SINR_j(MF) = \frac{A_j^2 N^2}{\sum_{k=1,k\neq j}^{K} A_k^2 |\mathbf{a}_j^H \mathbf{a}_k|^2 \ + \ 2\sigma^2 N} \tag{13}$$

With our definition of array response, $\mathbf{a}_j^H \mathbf{a}_j = N$. Note that the strength of the interference terms depends on terms like $\mathbf{a}_j^H \mathbf{a}_k = \mathbf{a}^H(\Omega_j)\mathbf{a}(\Omega_k)$ $(k \neq j)$, the correlation between the array responses at different spatial frequencies. We will explore the properties of such correlations, and how they depend on the number of antenna elements, later. But it is clear from (13) that, since such correlations are typically nonzero, the interference at the output of the spatial matched filter will lead to a performance floor.

4.2 MMSE Beamformer

The linear MMSE detector for user j e $\mathbf{c}_j$ minimizes the mean squared error (MSE)

$$MSE_j = E\left[|\mathbf{c}^H \mathbf{r}[n] - b_j[n]|^2\right] \tag{14}$$

The solution to this is given by

$$\mathbf{c}_j = \mathbf{R}^{-1}\mathbf{p}_j \tag{15}$$

where

$$\mathbf{R} = E[\mathbf{r}[n]\mathbf{r}^H[n]] \ , \quad \mathbf{p}_j = E[b_j^*[n]\mathbf{r}[n]] \tag{16}$$

The corresponding MMSE is given by

$$MMSE_j = E[|b_j[n]|^2] - \mathbf{p}_j^H \mathbf{R}^{-1}\mathbf{p}_j = 1 - \mathbf{p}_j^H \mathbf{R}^{-1}\mathbf{p}_j \tag{17}$$

assuming that the symbols are normalized to unit energy.

Adaptive implementation via least squares:. In a least squares implementation, we replace the statistical averages in (14)-(16) by empirical averages, assuming that we know N_t training symbols for the user. Thus, we seek to minimize (we use the same notation as for the statistical averages above for simplicity)

$$MSE_j = \frac{1}{N_t}\sum_{n=1}^{N_t} |\mathbf{c}^H \mathbf{r}[n] - b_j[n]|^2 \tag{18}$$

$$\widehat{\mathbf{c}}_j = \widehat{\mathbf{R}}^{-1}\widehat{\mathbf{p}}_j \tag{19}$$

where

$$\widehat{\mathbf{R}} = \frac{1}{N_t}\sum_{n=1}^{N_t} \mathbf{r}[n]\mathbf{r}^H[n] \ , \quad \widehat{\mathbf{p}}_j = \frac{1}{N_t}\sum_{n=1}^{N_t} b_j^*[n]\mathbf{r}[n] \tag{20}$$

Remarks:
(1) The computation of $\widehat{\mathbf{R}}$ and $\widehat{\mathbf{R}}^{-1}$ is common for all users, and does not require known training symbols.

(2) The computation of $\widehat{\mathbf{p}}_j$ requires a training sequence for user j, and must be done separately for each user.

Analytical insights into the MMSE beamformer: For our model (4), since the symbol sequences are normalized to unit energy and are uncorrelated, we obtain that

$$\mathbf{R} = \sum_{k=1}^{K} \mathbf{u}_k \mathbf{u}_k^H + 2\sigma^2 \mathbf{I} = \sum_{k=1}^{K} A_k^2 \mathbf{a}_k \mathbf{a}_k^H + 2\sigma^2 \mathbf{I} \tag{21}$$

$$\mathbf{p}_j = \mathbf{u}_j = A_j e^{j\phi_j} \mathbf{a}_j \tag{22}$$

so that the MMSE decision statistic for user j can be written as

$$Z_j^{MMSE}[n] = \mathbf{u}_j^H \mathbf{R}^{-1} \mathbf{r}[n] = A_j e^{-j\phi_j} \mathbf{a}_j^H \mathbf{R}^{-1} \mathbf{r}[n] \tag{23}$$

It can be shown that the MMSE beamformer for user j is equivalent to (a scalar multiple of) a whitened matched filter:

$$\mathbf{c}_j \sim \mathbf{R}_{\mathbf{I}_j}^{-1} \mathbf{u}_j = \left(\sum_{k=1, k \neq j}^{K} \mathbf{u}_k \mathbf{u}_k^H + 2\sigma^2 \mathbf{I} \right)^{-1} \mathbf{u}_j \tag{24}$$

where the matrix being inverted above is the covariance of the interference plus noise seen by user j. It is this inversion that attenuates strong interference.

The SINR for the MMSE correlator is given by

$$SINR_j(MMSE) = \frac{|\mathbf{c}_j^H \mathbf{u}_j|^2}{\sum_{k=1, k \neq j}^{K} |\mathbf{c}_j^H \mathbf{u}_k|^2 \ + 2\sigma^2 ||\mathbf{c}_j||^2}. = \frac{|\mathbf{c}_j^H \mathbf{u}_j|^2}{\mathbf{c}_j^H \mathbf{R}_{\mathbf{I}_j} \mathbf{c}_j} = \frac{1}{MMSE_j} - 1 \tag{25}$$

It can be shown that this simplifies to the SINR of a whitened matched filter:

$$SINR_j(MMSE) = \mathbf{u}_j^H \mathbf{R}_{\mathbf{I}_j}^{-1} \mathbf{u}_j = \mathbf{u}_j^H \left(\sum_{k=1, k \neq j}^{K} A_k^2 \mathbf{a}_k \mathbf{a}_k^H + 2\sigma^2 \mathbf{I} \right)^{-1} \mathbf{u}_j \tag{26}$$

5 Lab Assignment

0) Suppose each mobile has a M-element antenna array. If $P_{element}$ is the transmit power from a single antenna element, then the net transmit power is $MP_{element}$ (that is, we get a power pooling gain). In addition, if the mobile transmitter can form a beam towards the base station, then there is a beamforming gain of M. Thus, transmit beamforming leads to a net gain of M^2 compared to the power of an individual element. Based on this discussion, compute the power per transmit element (in dBm) required to attain an SNR of 15 dB after receive beamforming at an N-element base station, under the following assumptions:

- Carrier frequency 140 GHz
- Bandwidth 5 GHz (so QPSK yields 10 Gbps per user, not accounting for excess bandwidth!)
- Range 100m
- Receiver noise figure 8 dB
- $M = 8$, $N = 128$

1) Consider an N-element linear array with inter-element spacing of $\frac{\lambda}{2}$, and a field of view $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$.
(a) What is the range of spatial frequencies Ω as θ varies?
(b) For two different spatial frequencies Ω_1 and Ω_2, define the normalized correlation

$$\rho = \frac{|\mathbf{a}^H(\Omega_1)\mathbf{a}(\Omega_2)|}{||\mathbf{a}(\Omega_1)||||\mathbf{a}(\Omega_2)||} = \frac{|\mathbf{a}^H(\Omega_1)\mathbf{a}(\Omega_2)|}{N} \tag{27}$$

noting that $||\mathbf{a}(\Omega_1)||^2 = ||\mathbf{a}(\Omega_2)||^2 = N$. Show that ρ simplifies to a function of $\Delta\Omega = \Omega_1 - \Omega_2$ alone, given by

$$\rho(\Delta\Omega) = \left| \frac{\sin(N\Delta\Omega/2)}{N\sin(\Delta\Omega/2)} \right| \tag{28}$$

When beamforming towards a desired signal from direction Ω_1, if we normalize the desired signal amplitude to one, then $\rho(\Delta\Omega)$ is the relative amplitude of an interfering signal from direction Ω_2.
(c) For the model (4), show that the SINR at the output of the spatial matched filter for user j, given by (13), can be rewritten as

$$SINR_j(MF) = \frac{1}{\sum_{k=1,k\neq j}^{K} \frac{A_k^2}{A_j^2}\rho^2(\Omega_j - \Omega_k) \ + \ \frac{1}{SNR_j}} \tag{29}$$

Thus, $\rho^2(\Omega_j - \Omega_k)$ and the relative received powers $\frac{A_k^2}{A_j^2}$ determine the impact of interferer k on the performance for user j.

2) Let us now examine how ρ varies with N, the number of antennas.
(a) Plot $\rho(\Delta\Omega)$ over the range of $\Delta\Omega$ when the spatial frequencies vary as in (a), for $N = 16$ and $N = 256$.
Comment on the impact of N on the shape of the function.
(b) Redo the plot in dB. That is, plot $20\log_{10}\rho(\Delta\Omega)$.

3) Now, consider the analogous function in terms of AoA θ. Reusing notation, define

$$\rho(\theta_1, \theta_2) = \frac{|\mathbf{a}^H(\theta_1)\mathbf{a}(\theta_2)|}{||\mathbf{a}(\theta_1)||\,||\mathbf{a}(\theta_2)||} = \frac{|\mathbf{a}^H(\theta_1)\mathbf{a}(\theta_2)|}{N} \tag{30}$$

This can be obtained by plugging in the expression for spatial frequency in terms of AoA into (28). We call this a normalized beam pattern, obtained as a function of θ_2 when beamforming in direction θ_1.
(a) For a small difference $\Delta\theta = \theta_2 - \theta_1$, how does $\Delta\Omega$ depend on $\Delta\theta$ as a function of the beam direction θ_1? Do you see why trying to form a beam towards $\theta_1 = \frac{\pi}{2}$ is a bad idea?
Remark: The preceding consideration motivates us to restrict the field of view to smaller than $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Another reason is that individual antenna elements are not truly omnidirectional, providing gain over a limited angular spread. The choice of field of view to $[-\frac{\pi}{3}, \frac{\pi}{3}]$ for our system is sensible for antenna arrays composed of typical patch elements.
(a) For $N = 16$ and $N = 256$, plot $\rho(\theta_1, \theta_2)$ as a function of θ_2 (in degrees) for $\theta_1 = 0$ degrees (beamforming towards a user at broadside) and $\theta_1 = 60$ degrees (beamforming towards a user at the edge of the field of view).
(b) Redo a dB plot $20\log_{10}\rho(\theta_1, \theta_2)$.
(c) Comment on any qualitative differences you see between the beam pattern for $\theta_1 = 0$ degrees and $\theta_1 = 60$ degrees, and how it is predicted by 3(a). Which one is sharper around its peak?

Now, let us take the first step in setting up a simulation model for the system (4). There are K users, assumed to be uniformly distributed over an area defined by minimum and maximum ranges, R_{min} and R_{max}, respectively, from the base station, and the field of view $[-\frac{\pi}{3}, \frac{\pi}{3}]$. The link budget is such that the SNR of a "cell edge" user at maximum distance R_{max} is set to some desired value, which we denote by SNR_{edge}. Power control is not employed, with users transmitting at the same power. Thus, for free space propagation with received power scaling as $1/R^2$, the ratio of received powers for two users at distance R_1 and R_2 from the base station is given by

$$\frac{A_1^2}{A_2^2} = \frac{R_2^2}{R_1^2}$$

Define the load factor $\beta = \frac{K}{N}$ as the ratio of number of users to number of base station antennas.

The inputs to our system simulation are the parameters N, β, R_{min}, R_{max} and SNR_{edge} (and of course, the signaling constellation employed). Write your programs for general values of these parameters, but perform evaluations for $N = 128$, $R_{min} = 5$m, $R_{max} = 100$m, and $\beta = \frac{1}{4}$.

4) Write a program for generating a random draw of user locations $\{(R_k, \theta_k), k = 1, ..., K\}$ for $R_{min} = 5$m, $R_{max} = 100$m, $\theta_k \in [-\frac{\pi}{3}, \frac{\pi}{3}]$.
(a) Show that the cumulative distribution function (CDF) of the range R for a typical user is given by

$$P[R \leq r] = F_R(r) = \begin{cases} 0, & r < R_{min} \\ \frac{r^2 - R_{min}^2}{R_{max}^2 - R_{min}^2}, & R_{min} \leq r \leq R_{max} \\ 1, & r \geq R_{max} \end{cases} \tag{31}$$

(b) A random variable R with desired CDF $F_R(r)$ from a uniform random variable $U \sim Unif[0,1]$ by applying the function $R = F_R^{-1}(U)$. (This is an standard result that is very useful for simulations, and I encourage you to look up its derivation if you do not know it already.). For our specific scenario, show that you can generate R from $U \sim Unif[0,1]$ as follows:

$$R = \sqrt{(1-U)R_{min}^2 + UR_{max}^2}$$

(d) Now, do random draws of range and angle (you need two uniform random variables for each user), and plot the user locations (in polar coordinates: meters and degrees) for a random draw for $N = 128$, $\beta = \frac{1}{4}$.
Remark: If two randomly drawn angles of arrivals are "too close" (e.g., within 2 degrees), you might want to throw away the samples and draw again. In practice, this means that the base station would schedule users who are "too close" in different time slots.

5) Setting $\sigma^2 = 1$ for convenience, and for a specified value of SNR_{edge} (dB), augment your program by assigning amplitudes $\{A_k, k = 1, ..., K\}$ to the users. The amplitude A_{edge} for a user at the edge R_{max} satisfies $SNR_{edge} = \frac{NA_{edge}^2}{2\sigma^2}$. Thus, the amplitude for any user at range R is given by $A = A_{edge}\frac{R_{max}}{R}$. Plot, analytically or by simulations, a histogram of relative powers $\frac{A_k^2}{A_{edge}^2}$ (in dB) for $R_{min} = 5$m, $R_{max} = 100$m.

6) For a given draw of user locations and amplitudes, compute the SINR obtained by spatial matched filtering for the *nearest and furthest users* using the analytical formula (13), setting SNR_{edge} to 15 dB.

7) Repeat 6) when you use the (analytically computed) MMSE receiver for each of these two users, and compare the SINR performance with that for spatial matched filter.

8) Plot the normalized beam pattern in dB of the MMSE correlator for the furthest user; i.e., compute

$$G(\theta)(dB) = 20\log_1 0\left(\frac{|\mathbf{c}^H\mathbf{a}(\theta)|}{||\mathbf{c}||||\mathbf{a}(theta)||}\right)$$

and plot against $\theta \in (-\pi/3, \pi/3)$. Mark the locations of the nearest and furthest users in your plot. You should see a strong gain towards the furthest user and a null towards the nearest user.
Remark: The MMSE receiver for any given user should steer nulls towards all the other users, so you should see nulls towards all users other than the furthest one.

9) For QPSK with Gray coding, if we model the noise plus interference at the output of a correlator as complex Gaussian, then the BER can be approximated by $Q\left(\sqrt{SINR}\right)$. For the analytically computed MMSE receivers in 7), estimate the BER for the nearest and further users.

10) Verify the BER estimates in 8) for the furthest user by simulating the performance of the analytically computed MMSE receiver by generating random symbols for the model (4). Make sure you simulate over enough symbols to obtain an accurate estimate.

11) Now, do an adaptive implementation of the MMSE correlator for the furthest user, assuming that you have a training sequence of 2000 randomly drawn QPSK symbols for that user. Compute the SINR for the correlator you obtain using (25).

12) For the MMSE correlator obtained in 10), compare the BER estimate using $Q\left(\sqrt{SINR}\right)$ against simulations. How does this BER performance compare against the BER performance of the analytically computed MMSE receiver in 9)?

13) **optional** Estimate analytically or by simulation the BER of a windowed beamspace MMSE receiver (to be discussed in lecture) for the furthest user.

6 Lab Report

- Discuss the results you obtain, answer any specific questions that are asked, and print out the most useful plots to support your answers.
- Append your programs to the report. Make sure you comment them in enough detail so they are easy to understand.

- Write a paragraph about any questions or confusions that you may have experienced with this lab.