Acquisition in Direct-Sequence Spread-Spectrum Communication Networks: An Asymptotic Analysis

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***Abstract*—If direct-sequence spread-spectrum (DS/SS) modulation is employed, the receiver must acquire the chip timing for incoming messages before these messages can be demodulated. If the system or network employs DS/SS multiple access, also known as code-division multiple access (CDMA), the acquisition process is more difficult because multiple messages are transmitted simultaneously and in the same frequency band. The effect of multiple-access interference on the acquisition of DS/SS signals is studied. A passive matched filter approach is employed, and the acquisition window length, which is the length of the matched filter, determines the complexity of the acquisition scheme. The acquisition-based capacity of a DS/SS system is defined to be the maximum number of simultaneous transmissions permissible while maintaining acceptable acquisition performance. The performance of the acquisition scheme is evaluated for large acquisition window length, and the asymptotic analysis yields the capacity as a function of the acquisition window length. If this length is linearly related to the processing gain, the acquisition-based capacity is smaller than that obtained by consideration of post-acquisition criteria only (e.g., bit-error probability for the demodulated signal). The results indicate the relative importance of the acquisition problem and suggest directions for further research.**

***Index Terms*—Code-division multiple access (CDMA), Direct-sequence spread spectrum, Acquisition**

I. Introduction

DIRECT-SEQUENCE spread-spectrum (DS/SS) multiple access, also known as code-division multiple access (CDMA), has attracted much recent interest for multiple-access communications in commercial applications such as digital cellular radio and personal communication networks. In a DS/SS scheme, each data signal to be transmitted is spread over a much larger bandwidth using an appropriately chosen sequence. The sequence for a particular transmitter is referred to as the *spreading sequence* or *signature sequence* for that transmitter. An appropriate selection of the set of signature sequences provides the cross correlation properties that enable several DS/SS transmissions to share the same channel [13]. We are interested in characterizing the *capacity* of DS/SS communications, which is defined to be the maximum number of simultaneous transmissions that the system can support, expressed as a function of some measure of the complexity of the scheme.

We focus our attention on the problem of acquiring a DS/SS signal in the presence of multiple-access interference. Our goal is to characterize the *acquisition-based capacity*, which is defined to be the maximum number of simultaneous transmissions supported by the DS/SS scheme while maintaining acceptable acquisition performance. A passive matched filter approach [5] is employed for acquisition. The length of the matched filter, referred to as the *acquisition window length*, determines the complexity of the acquisition scheme. The acquisition-based capacity is the maximum number of simultaneous transmissions as a function of the acquisition window length N, and our main result is that this capacity is of the order of $N/\log N$.

The usual approach [6], [9]–[10], [12] taken in the analysis of DS/SS schemes is to consider the bit error rate (BER) or the signal-to-noise ratio (SNR) as the measures of performance, and the *bandwidth expansion* N_B (sometimes termed the processing gain) is employed as the measure of the complexity of the scheme. The capacity based on the BER or SNR criteria is known to have a linear dependence on N_B (e.g., [10], [12]). This result assumes implicitly that the receiver has already acquired the desired signal, and is therefore able to despread it using the appropriate signature sequence. If the acquisition window length is a linear function of N_B, our result indicates that the acquisition-based capacity is lower than the one derived using BER or SNR criteria. Furthermore, numerical results indicate that our asymptotic formula significantly overestimates the acquisition-based capacity when N is relatively small. Thus, the acquisition problem may ultimately limit the capacity of DS/SS schemes. It is worth stressing that we assume that the only signals that are provided to the acquisition systems are the DS/SS signals themselves; no auxiliary timing signals are employed. In fact, one conclusion that can be drawn from our results is that, in order to achieve high capacity in a practical application, more *a priori* timing information than is assumed in this paper is needed. Such information could be provided by means of an interference-free side channel, for instance.

The system model, including the description of an acquisition scheme based on a threshold rule, is presented in

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Section II. The derivation of an asymptotic formula for the acquisition-based capacity and some numerical results illustrating convergence issues are given in Section III. In Section IV, we show that no asymptotic advantage in performance is obtained by considering a more complicated acquisition scheme that is based on locating the maximum of the sequence of matched filter outputs. Our conclusions are presented in Section V.

II. System Model

The model for a DS/SS signal $s(t)$ is the standard model used in the literature (see [6], [9]–[10], [12]). The received signal for a typical transmission is

$$s(t) = \sqrt{2P}x(t-\tau_p)d(t-\tau_p)\cos(\omega_c t + \phi_c),$$

where $x(\cdot)$ is the spreading waveform, $d(\cdot)$ is the data signal, ω_c is the carrier frequency, τ_p is the propagation delay, ϕ_c is the uncertainty in the carrier phase, and P is the signal power. The spreading waveform $x(t)$ is taken to be a doubly infinite sequence of rectangular pulses of duration T_c. The amplitude of the nth pulse is denoted by x_n, and x_n is either $+1$ or -1. The sequence (x_n) is the signature sequence. The data signal is a sequence of rectangular pulses of duration T, where T is a multiple of T_c. Thus, if $T = N_B T_c$, then there are N_B signature pulses, or *chips*, in each data pulse.

Acquisition [5] refers to the task of obtaining an estimate of the delay of a given transmission of interest, which will henceforth be referred to as the *target transmission*. In order to focus our attention on how the acquisition problem (as distinct from the problem of data transmission) limits the system capacity, we assume that there is no data modulation on the target transmission. Inclusion of data modulation would only decrease our estimates of the acquisition-based capacity. Furthermore, if a special preamble is used for the purpose of acquisition, there is no data modulation on this part of the transmission. The received signal due to the target transmission is thus given by

$$r_0(t) = \sqrt{2}\left(\sum_{j=-\infty}^{\infty} a_j p_{T_c}(t - jT_c - \tau T_c)\right)\cos(\omega_c t + \phi),$$

where p_{T_c} is a unit rectangular pulse of duration T_c, (a_j) is the signature sequence for the target transmission, and the power of the target transmission is normalized to one. For the model we have adopted, the delay τ can be restricted to the range $0 \le \tau \le N-1$. The net received signal $r(t)$ is given by

$$r(t) = r_0(t) + X(t),$$

where $X(t)$ is additive noise and interference. We consider only the effect of multiple-access interference in this paper, and we assume that this interference consists of J other DS/SS transmissions at the same carrier frequency. The jth interfering transmission is given by

$$r_j(t) = \sqrt{2P_j}\sum_{k=-\infty}^{\infty} x_k^{(j)} p_{T_c}(t - kT_c - \tau_j T_c) \cdot \cos(\omega_c t + \phi + \theta_j), \qquad 1 \le j \le J,$$

where P_j is the power relative to that of the target transmission and θ_j is the phase relative to that of the target transmission. The sequence $(x_k^{(j)})$ results from a multiplication of the data and signature sequences of the interfering transmission. Our analysis considers random signature sequences (this is specified in more detail in the sequel), so that it suffices to consider τ_j to lie in the range $0 \le \tau_j \le 1$. The additive interference is thus given by

$$X(t) = \sum_{j=1}^{J} r_j(t).$$

It is assumed that the receiver can acquire the carrier frequency and the phase of the target transmission perfectly. This is a reasonable assumption for our purpose, which is to show that the acquisition problem seriously limits the capacity, even under such favorable conditions. We assume that the delay τ for the target transmission is an integer, enabling us to consider the following discrete-time model. The receiver computes the statistics

$$Z_k = \frac{\sqrt{2}}{T_c}\int_{kT_c}^{(k+1)T_c} r(t)\cos(\omega_c t + \phi)dt = a_{k-\tau} + X_k,$$

where the additive interference X_k is given by

$$X_k = \sum_{j=1}^{J} \sqrt{P_j}\cos\theta_j\left[(1-\tau_j)x_k^{(j)} + \tau_j x_{k-1}^{(j)}\right]. \tag{1}$$

The acquisition problem consists of estimating τ based on the sequence of statistics (Z_k).

We consider acquisition using a discrete-time filter matched to a section $a_{-N+1}, a_{-N+2}, \cdots, a_0$ of the signature sequence of the target transmission. The filter coefficients are given by $h_i = a_{-i}$ for $i = 0, 1, \cdots, N-1$. The acquisition window length N is a measure of the complexity of the acquisition scheme. If the input to the filter is a sequence (u_k), the output (y_n) is given by

$$y_n = \sum_{i=0}^{N-1} h_i u_{n-i}.$$

Thus, if the input sequence is a time-shifted version of the target signature sequence, given by $u_k = a_{k-\tau}$, then the filter output at time τ is N.

For a signature sequence with good autocorrelation properties, the output of the matched filter at other times is small. In addition, if the set of signature sequences used by the various transmissions on the network have good cross correlation properties, the contribution of an interfering transmission to the output of the matched filter is small. Thus, acquisition in the presence of interference and noise may be achieved by detecting when the matched filter output crosses a threshold. Specifically, we consider a threshold rule described in the following. Let (W_n) be the net output of the matched filter (due to both the target transmission and the interference). The delay estimate $\hat{\tau}$ is given by

$$\hat{\tau} = \min\{n \ge 0 : W_n/N > \alpha\}, \tag{2}$$

where $\alpha \in (0,1)$ is a threshold.

An alternative scheme is to consider the maximum of the matched filter output over a given time interval as a means of estimating the delay. Such a scheme performs better than the threshold rule (2), but is more complicated to implement. Further, it is shown in Section IV that there is no asymptotic advantage in performance to be gained by using a maximum-based scheme. In view of this result, we restrict attention in this section to an acquisition scheme that employs the threshold rule (2).

For the purpose of the analysis, we make some further assumptions. The true delay τ of the target transmission is modeled as a random variable that takes on each integer value in the interval $[0, N-1]$ with probability $1/N$. Actually, our results hold under the weaker assumption that τ is of the order of N. The signature sequences for all the transmissions are modeled as random, and the signature sequences for each of the transmissions heard by the receiver are assumed to be independent. For any given signature sequence (x_n), the elements x_n are assumed to be independent and identically distributed, taking the values $+1$ and -1 with equal probability. For the purpose of this paper, any such random variable is called a *symmetric Bernoulli* random variable, and the sum of J independent symmetric Bernoulli random variables is called a *symmetric binomial* random variable with parameter J.

We consider a one-shot acquisition problem in which the acquisition algorithm ignores the output of the matched filter subsequent to the first threshold crossing. The performance criterion considered is the probability of *acquisition failure*, which is said to occur if the first threshold crossing occurs at an incorrect delay, or if no threshold crossing occurs. A *false alarm* is said to occur if the threshold is exceeded before the correct instant (i.e., if $W_n/N > \alpha$ for some $n < \tau$), and a *miss* is said to occur if the threshold is not exceeded at the correct time (i.e., if $W_\tau/N < \alpha$). Acquisition failure occurs in the event of either a false alarm or a miss. The outputs of the matched filter after time τ do not affect the probability of acquisition failure in our model, since the acquisition algorithm considers these outputs only in the event of a miss, in which case acquisition failure has already occurred. Threshold crossings beyond time τ are therefore not included in the definition of false alarm. This is in contrast to acquisition algorithms with mechanisms for recovering from failure [7], [11], for which erroneous threshold crossings subsequent to a miss affect the recovery time from an acquisition failure, and hence must be accounted for.

The target transmission contributes a value N to the matched filter output W_τ at time τ. At other times $n < \tau$, its contribution to the matched filter output W_n is asymptotically negligible for the large values of N and J of interest. Hence, we simplify our analysis by setting these contributions to zero. This eliminates the need to consider the dependence between the target signature sequence and the matched filter coefficients. Let (Y_n) be the matched filter output due to the interference alone. We are assuming, then, that

$$W_n = Y_n, n < \tau; \qquad W_\tau = Y_\tau + N. \tag{3}$$

With the above simplification, the probability of false alarm P_{FA} is given by

$$P_{\text{FA}} = P\left[\cup_{n=0}^{\tau-1}\{Y_n/N > \alpha\}\right], \tag{4}$$

and the probability of miss P_{M} is given by

$$P_{\text{M}} = P[Y_\tau/N < -(1-\alpha)]. \tag{5}$$

The probability of acquisition failure P_{F} is the probability of the union of the events considered in (4) and (5). For our purpose, it suffices to bound it above and below as follows:

$$P_{\text{FA}} \leq P_{\text{F}} \leq P_{\text{FA}} + P_{\text{M}}. \tag{6}$$

While the approximation (3) does not affect our asymptotic results, it is worth pointing out that a better approximation for finite N and J may be to model the contribution of the target transmission at times $n < \tau$ as resulting from another independent interfering transmission.

An evaluation of capacity consists of finding the maximum permissible number of interfering signals J as a function of the acquisition window length N, subject to the constraint that the probability of acquisition failure tends to zero as $N \to \infty$.

III. Analysis

We concentrate on the false alarm probability P_{FA}, since it turns out that this is the quantity that determines the capacity. More precisely, the miss probability tends to zero for the maximum value of J such that the probability of false alarm tends to zero. Hence, using (6), the probability of acquisition failure also tends to zero for such J. We consider first a *synchronous* multiple-access system, in which the J interfering signals have the same power as the target transmission, and are both phase- and chip-synchronous with respect to the target transmission. That is,

$$P_j = 1, \theta_j = 0, \text{and } \tau_j = 0, \qquad \text{for } 1 \leq j \leq J.$$

For these assumptions, the matched filter input at time k due to the interference is given by (see (1))

$$X_k = \sum_{j=1}^{J} x_k^{(j)}. \tag{7}$$

We henceforth consider the matched filter input and output due to the interference alone, since we are ignoring the contribution of the target transmission at times $n < \tau$ (see (3)). We see from (7) that the input is a sequence of independent and identically distributed symmetric binomial random variables with parameter J. The matched filter coefficients themselves are independent and identically distributed symmetric Bernoulli random variables, and are independent of the input sequence.

Conditioned on the delay τ (assume $\tau \geq 1$ here), the probability of false alarm is given by

$$P_{\text{FA}}(\tau) = P\left[\cup_{n=0}^{\tau-1}\{Y_n/N > \alpha\}\right].$$

Although the matched filter input is a sequence of independent random variables, the matched filter output is not, so that the above expression is not easy to evaluate exactly. Instead, we develop upper and lower bounds on P_{FA} to obtain our asymptotic result, which is that the capacity is given by $\frac{1}{2}\ \alpha^2\ (N/\log N)$.

Using a union bound and the first two terms of the inclusion-exclusion principle, we can obtain upper and lower bounds on the false alarm probability as follows:

$$P_{\mathrm{FA}}(\tau) \leq \tau P[Y_0/N > \alpha], \tag{8}$$

$$P_{\mathrm{FA}}(\tau) \geq \tau P[Y_0/N > \alpha] - \sum_{n=0}^{\tau-1} \sum_{m=n+1}^{\tau-1} P[Y_m/N > \alpha, Y_n/N > \alpha], \tag{9}$$

where we have used the fact that the Y_n are identically distributed.

We now proceed with the asymptotic evaluation of the upper bound (8). The matched filter output at a given time (say zero) is given by

$$Y_0 = \sum_{i=0}^{N-1} \sum_{j=1}^{J} h_i x_{-i}^{(j)},$$

which is the sum of NJ independent and identically distributed symmetric Bernoulli random variables. By the central limit theorem, $(NJ)^{-1/2}\ Y_0$ tends in distribution to a standard Gaussian random variable as the product NJ gets large. Thus, the upper bound (8) is asymptotically evaluated as

$$\tau P\left[(NJ)^{-1/2} Y_0 > \alpha(N/J)^{1/2}\right] \approx \tau Q\left[\alpha(N/J)^{1/2}\right], \tag{10}$$

where $Q(\cdot)$ is the complementary standard Gaussian distribution function defined as

$$Q(x) = \int_x^\infty (2\pi)^{-1/2} \exp\left(-x^2/2\right) dx.$$

The random variable τ is uniformly distributed on the set of integers 0 through $N-1$. By averaging over τ in (10), we obtain

$$P_{\mathrm{U}} = (N-1)/2\ Q\left[\alpha(N/J)^{1/2}\right] \tag{11}$$

as an asymptotic upper bound on P_{FA}. Using the result [15, p. 39]

$$Q(x) \leq (2\pi)^{-1/2} x^{-1} \exp\left(-x^2/2\right),$$

it is readily seen that, if N is sufficiently large, then P_{U} (hence P_{FA}) can be made as small as desired provided

$$J = \tfrac{1}{2}\alpha^2 (N/\log N)(1-\delta)$$

for any $0 < \delta < 1$. Thus, the maximum allowable J must be strictly less than $\frac{1}{2}\ \alpha^2\ (N/\log N)$, which is therefore a lower bound on the capacity.

We should note that, as just presented, the application of the standard form of the central limit theorem for evaluating the upper bound on P_{FA} is naive, since, for J of the order of $N/\log N$, the argument of $Q(\cdot)$ in the asymptotic expression on the right hand side of (11) tends to infinity as $N \to \infty$. However, the development can be made rigorous using a large deviations version of the central limit theorem [4, p. 549], which is restated in the following proposition.

Proposition 1: Let (U_k) be a sequence of independent and identically distributed random variables with mean zero and variance 1, and suppose that the moment generating function $M(s) = E\{\exp(sU_1)\}$ exists for all s in some interval $(-\epsilon, \epsilon)$ for $\epsilon > 0$. If $S_n = \sum_{k=1}^n U_k$, and if $x = o(n^{1/6})$, then

$$P\left[n^{-1/2} S_n > x\right] = Q(x)(1 + o(1)), n \to \infty.$$

In our application, the random variables being summed are the symmetric Bernoulli random variables $h_i x_{-i}^{(j)}$, which do satisfy the conditions of the proposition. We apply the proposition to obtain the asymptotic expression (10) from (8), setting $n = NJ$ and $x = (N/J)^{1/2}$. The condition $x = o(n^{1/6})$ specializes to the condition $N/J^2 \to 0$, and this is satisfied if J is of the order of $N/\log N$.

For an asymptotic evaluation of the lower bound (9), we have to approximate joint probabilities of the form $P[N^{-1}Y_m > \alpha,\ N^{-1}Y_n > \alpha]$ for $m \neq n$. As shown in the Appendix, each of these terms is asymptotically evaluated as $Q^2[\alpha(N/J)^{1/2}]$. This is done by expressing the random vector $(NJ)^{-1/2}(Y_m, Y_n)$ as a normalized sum of suitably chosen independent (but not identically distributed) random vectors, and using a large deviations version of a multivariate central limit theorem [1], [14]. Replacing the terms in (9) by their asymptotic estimates, and assuming, as before, that τ is uniform over the integers in $[0, N-1]$, we obtain

$$P_{\mathrm{L}} = (N-1)/2\ Q\left[\alpha(N/J)^{1/2}\right] - 1/6\ (N-1)(N-2) Q^2\left[\alpha(N/J)^{1/2}\right] \tag{12}$$

as an asymptotic lower bound on P_{FA}.

Let $f(r) = r - 2/3\ r^2$. From (11) and (12), $P_{\mathrm{L}} \geq f(P_{\mathrm{U}})$. Now, the function $f(r)$ increases (from 0 to 3/8) with r for $r \in [0, 3/4]$. From [15, p. 39], we have

$$Q(x) \geq (2\pi)^{-1/2} x^{-1}\left(1 - x^{-2}\right) \exp\left(-x^2/2\right).$$

Substituting in (11), we find that for N sufficiently large, we can make $P_{\mathrm{U}} \geq r$ for any $r \in [0, 3/4]$ if

$$J = \alpha^2/2 (N/\log N)(1 + \delta) \tag{13}$$

for any $\delta > 0$. Thus, we can make P_{U} tend to 3/4 from below for such a choice of J, which implies that $f(P_{\mathrm{U}})$ tends to 3/8 from below. Since $P_{\mathrm{L}} \geq f(P_{\mathrm{U}})$, we obtain that P_{L}, and hence P_{FA}, is asymptotically bounded away from zero if J satisfies (13). However, the asymptotic lower bound (12) becomes trivial for larger J; for instance, $J = N$ implies that $P_{\mathrm{L}} \to -\infty$ as $N \to \infty$. In order to circumvent this difficulty, we need the following proposition, which is proved in the Appendix.

Proposition 2: Let P_{FA} be the false alarm probability corresponding to J interfering transmissions. Let P'_{FA} be the false alarm probability if, in addition to these transmissions, there are other interfering transmissions. The input to the matched filter due to any given interfering transmission is a sequence of independent, zero-mean, symmetric random variables, and input sequences corresponding to different interfering transmissions are independent. Then, $P'_{\mathrm{FA}} \geq \frac{1}{2}\ P_{\mathrm{FA}}$.

Intuitively, we expect that the false alarm probability increases with the number of interfering transmissions, but we have not been able to prove that. Proposition 2 is a weaker result, but it suffices for our purpose, which is to conclude that $P_{\rm FA}$ is bounded away from zero for J larger than in (13). This yields that $\frac{1}{2}\alpha^2 (N/\log N)$ is an upper bound on the capacity, which is the same as the lower bound previously obtained.

Letting $\alpha \rightarrow 1$ from below, we obtain that the acquisition-based capacity for synchronous DS/SS CDMA is $\frac{1}{2}(N/\log N)$. It remains to address the following technical detail in order to complete the above argument. As long as $\alpha < 1$, the miss probability $P_{\rm M} \rightarrow 0$ as $N \rightarrow \infty$; setting $\alpha = 1$, however, yields $P_{\rm M} = \frac{1}{2}$. Thus, we must let $\alpha \rightarrow 1$ as a function of N in such a manner that the the miss probability tends to zero as $N \rightarrow \infty$. An asymptotic analysis of the expression for $P_{\rm M}$ shows that the required condition is that $\alpha \rightarrow 1$ in such a way that

$$(1-\alpha)^2 \log N \rightarrow \infty. \tag{14}$$

Our result can be generalized to *asynchronous* multiple-access systems with unequal power signals. In such systems, the interfering transmissions are not phase- or chip-synchronous, and the additive interference X_k has the the more general form given in (1). The phases θ_j for the interfering transmissions are modeled as independent random variables, uniformly distributed over $[0, 2\pi]$. The delays τ_j take their values in the interval $[0,1]$, but no assumption on their distribution is needed. The capacity is a function of $\bar{P}$, the average interference power relative to the power in the target signal. Since the target signal has unit power, $\bar{P} = J^{-1}\sum_{j=1}^{J} P_j$, where P_j is the power in the j-th interfering transmission. Upper and lower bounds on the capacity are derived in the Appendix. The lower bound on capacity remains essentially unchanged from that for the synchronous system, being given by $\frac{1}{2}\alpha^2(N/\log N)/\bar{P}$. The upper bound, given by $2\alpha^2(N/\log N)/\bar{P}$, is a factor of four larger than the lower bound (a factor of two due to asynchronism in phase, and a factor of two due to asynchronism in chip alignments). This is unlike the synchronous system, for which the upper and lower bounds are asymptotically the same. It may be possible to find tighter asymptotic bounds, but our concern here is with the dependence of the capacity on the acquisition window length; small constant factors are not important.

It is interesting to note the similarity in the form of our results to that of previously derived results on the capacity of neural networks [8]. The problem addressed in [8] is roughly stated as follows. It is desired to store m binary vectors of length n (called memories) in a neural network constructed using an outer-product construction. The output corresponding to any given input to the neural network should be the memory that is closest to the input vector. This issue is treated in a probabilistic setting in [8]: the m memories are assumed to be randomly chosen, and it is desired to determine the capacity of the corresponding neural network, which is defined to be the maximum value of m as a function of n such that the memories are recovered with high probability from a noisy input. This capacity is found to be of the order of $n/\log n$. The similarity of our results to those in [8] is not accidental, in that the mathematical problems in the two situations, although not identical, are very similar. In both problems, the inclusion-exclusion principle and large deviations versions of the central limit theorem are employed as a means of estimating the probabilities of interest. The structure of the problem in [8] permits a more detailed analysis than is possible (or necessary) in our application. The large deviations results [1], [4, p. 549], [14] used in this paper are more general in scope than those used in [8], and it is possible that, together with an analog of Proposition 2, they could be used to simplify the derivation of the results in [8]. Pursuing these issues, however, is beyond the scope of this paper.

We conclude this section with a discussion of convergence issues. The asymptotic capacity estimate $\frac{1}{2}(N/\log N)$ relies on letting the threshold $\alpha \rightarrow 1$. However, as indicated by the condition (14), for finite N, the threshold α may have to be significantly less than one in order to keep the miss probability small. The following numerical example illustrates this point. Consider $N = 63$, and the following approximation to $P_{\rm F}$ based on a union bound and the central limit theorem:

$$P_{\rm F} \approx \tfrac{1}{2}(N-1)\, Q\left[\alpha(N/J)^{1/2}\right] + Q\left[(1-\alpha)(N/J)^{1/2}\right].$$

For each value of J, let $\alpha(J)$ denote the value of α that minimizes the above probability, and let the resulting probability be denoted as $P^*(J)$. For a desired acquisition failure probability of $P^*(J) \geq 10^{-2}$, we obtain $J = 1$ (the corresponding best threshold is $\alpha(J) = 0.55$) as the maximum allowable number of interfering transmissions. In contrast, the asymptotic capacity formula $1/2(N/\log N)$ yields the optimistic estimate $J = 7$. It is worth comparing these numerical results to the capacity obtained using a post-acquisition criterion such as the signal-to-interference ratio (SIR). For a synchronous CDMA system with bandwidth expansion N_B, the SIR is given by N_B/J. For $N_B = 63$ and a desired SIR of 10 dB, the maximum allowable number of interfering transmissions $J = 6$. Thus, for small N and N_B, post-acquisition criteria may yield a smaller capacity than the *asymptotic* formula for acquisition-based capacity; however, a more careful calculation of the latter indicates that acquisition performance limits the capacity for relatively small N as well.

Assuming that the threshold α is small enough to attain an acceptable miss probability, we must now examine the rate at which the false alarm probability $P_{\rm FA}$ approaches zero as $N \rightarrow \infty$ in order to obtain results for finite N. Consider the approximation

$$P_{\rm FA} \approx \tfrac{1}{2}(N-1)\, Q\left[\alpha(N/J)^{1/2}\right],$$

and let $J = 1/2\,\alpha^2\,(N/\log N)\,(1-\delta)$. It is easy to show that an upper bound on the value of N required to achieve a desired value of $P_{\rm FA}$ is given by the transcendental equation

$$N = \left(\frac{\sqrt{1-\delta}}{4P_{\rm FA}\sqrt{\pi \log N}}\right)^{(1-\delta)/\delta}.$$

For $P_{\rm FA} = 10^{-2}$, this equation yields $N = 1.83 \times 10^5$ for $\delta = 0.1$, and $N = 61$ for $\delta = 0.3$. The asymptotic capacity formula $\frac{1}{2}\alpha^2 (N/\log N)$ is thus optimistic, in that it is obtained only within a factor of $(1-\delta)$, which may be significantly less than one for finite N. In general, the smaller the value of N, the larger the value of δ required to attain a given value of P_{FA}. The foregoing discussion illustrates that the acquisition-based capacity for finite N is *less* than that predicted by our asymptotic formulas, which reinforces our contention that the acquisition problem may be a significant limiting factor in the capacity of CDMA systems.

IV. Comparison with a Maximum-Based Acquisition Scheme

We compare the threshold rule considered in our analysis to a *maximum-based* acquisition scheme that forms a delay estimate based on the maximum of the matched filter output sequence over a given time interval. It is assumed, as before, that the true delay τ is an integer in the interval $[0, N-1]$. The delay estimate $\hat{\tau}_{\max}$ for this scheme is given by

$$\hat{\tau}_{\max} = \arg \max_{0 \le n \le N-1} W_n, \tag{15}$$

where W_n is the matched filter output at time n.

We first show that, asymptotically as $N \to \infty$, the maximum-based scheme performs as well as any threshold scheme. As remarked in Section III, in order for the miss probability to tend to zero as $N \to \infty$, the threshold α must be strictly less than one. Since the capacity expression for the threshold scheme increases with the relative threshold α, a lower bound for the capacity of the maximum-based scheme is obtained by letting $\alpha \to 1$ (from below) in the expression for the lower bound on capacity for a threshold scheme. For the maximum-based scheme, therefore, $\frac{1}{2}\ (N/\log N)$ is a lower bound on capacity for a synchronous system, and $\frac{1}{2}\ (N/\log N)/\bar{P}$ is a lower bound on capacity for an asynchronous system.

Upper bounds on capacity for the maximum-based scheme are derived by showing that, for any $\beta > 0$, the probability of acquisition failure for the maximum-based scheme is asymptotically the same as the false alarm probability for a *fictitious* threshold scheme with threshold $\alpha = 1+\beta$. Although it is not possible to employ such a threshold scheme in practice due to unacceptable miss probabilities, this consideration does not affect the asymptotic analysis of the false alarm probability for such a threshold scheme, so that we can substitute $\alpha = 1+\beta$ in the upper bounds on capacity for the threshold scheme to obtain corresponding upper bounds on capacity for the maximum-based scheme (for each $\beta > 0$). Letting $\beta \to 0$ (from above), we obtain that, for the maximum-based scheme, $\frac{1}{2}\ (N/\log N)$ is an upper bound on capacity for a synchronous system, and $2(N/\log N)/\bar{P}$ is an upper bound on capacity for an asynchronous system.

For proving the results previously stated, it is convenient to assume that the true delay $\tau = N-1$. The proof under the assumption that τ is $O(N)$ is quite similar, and is omitted. From (15), we see that, for the maximum-based scheme, acquisition failure occurs if and only if the matched filter output at time $n \le N-2$ exceeds the output at the true delay $N-1$; that is,

$$\max_{0 \le n \le N-2} W_n > W_{N-1}. \tag{16}$$

If a threshold scheme processes a sequence of matched filter outputs satisfying (16), there are two possible results. Either the threshold is not exceeded by any of the outputs (which results in a miss), or, by virtue of (16), the threshold is exceeded at time $n \le N-2$ (which results in a false alarm). Acquisition failure occurs in either situation, so that a threshold scheme always performs worse than the maximum-based scheme if the true delay τ assumes its maximum value. This yields lower bounds on the capacity of the maximum-based scheme as argued earlier. The foregoing argument can be modified quite easily to handle the more general situation when τ is $O(N)$ (uniformly distributed τ is a special case of the latter).

The probability of acquisition failure P_{F} for the maximum-based scheme is the probability of the event (16), which reduces, under the approximation (3), to

$$P_{\mathrm{F}} = P\left[\max_{0 \le n \le N-2} Y_n > Y_{N-1} + N\right].$$

For any $\beta > 0$, we have

$$\begin{aligned} P_{\mathrm{F}} &\ge P\left[\max_{0 \le n \le N-2} Y_n > Y_{N-1} + N | Y_{N-1} < \beta N\right] \\ &\quad \cdot P[Y_{N-1} < \beta N] \\ &\ge P\left[\max_{0 \le n \le N-2} Y_n > (1+\beta)N\right] P[Y_{N-1} < \beta N]. \end{aligned} \tag{17}$$

If the number of interfering transmissions J is of the order of $(N/\log N)$, it is easy to see that, for any $\beta > 0$, $P[Y_N - 1 < \beta N]$, the second term on the extreme right-hand side of (17), tends to one as $N \to \infty$. The first term on the extreme right-hand side can be bounded away from zero as $N \to \infty$ in exactly the same manner as the false alarm probability for a fictitious threshold scheme with threshold $\alpha = 1+\beta$. Using this value of α, we can bound P_{F} away from zero for J as in (13) for a synchronous system, and for J as in (A.9) for an asynchronous system. It is also easy to prove an analogue of Proposition 2 for the maximum-based scheme. Since these arguments hold for any $\beta > 0$, we obtain upper bounds on capacity by letting $\beta \to 0$.

We conclude, therefore, that using a threshold rule with threshold α close to one yields a performance that is asymptotically almost as good as that of the maximum-based scheme. Since the maximum-based scheme is more complex, a threshold scheme is preferable in practice.

V. Conclusion

The results of this paper show that the problem of acquisition imposes a limitation on the capacity of DS/SS networks. As discussed in Section I and Section III, the acquisition-based capacity result is more pessimistic than the the result based on the post-acquisition BER and SNR criteria, provided that the acquisition window length N is linearly related to the bandwidth expansion N_B. Despite some simplifying assumptions, we believe that our model captures the essential features of why the task of acquisition is so difficult. The basic reason is that, unless the receiver has a very good idea of the delay of the target transmission, there are too many opportunities for the interfering transmissions to produce a false alarm, and this results in a reduction of capacity by a factor of $\log N$. The reason a false alarm is so damaging in many applications is that it may cause the receiver to enter into useless and time-consuming demodulation, decoding, and tracking procedures that can lead to the loss of the opportunity to acquire the signal of interest. This is in contrast to the demodulation process, in which the interference can affect only one bit at a time (interleaving the bits if necessary), and error-control coding can be effectively exploited to yield reliable transmission. On the positive side, we must note that the performance of the acquisition scheme considered here (and indeed, that of any acquisition scheme) can be significantly improved by decreasing the initial timing uncertainty. For instance, if the timing uncertainty is of the order of a constant rather than of the order of N, it is easy to see that the acquisition-based capacity is of the order of N rather than $N/\log N$.

The random sequence model used here has been used in BER evaluations [6], [9]–[10], as well as in other models of acquisition [2]–[3]. Thus, we have considerable confidence in the qualitative nature of our conclusions, and we feel that the acquisition problem requires careful examination. We conclude by mentioning several issues that require further research.

If there are a large number of interfering transmissions, the acquisition window required for good acquisition performance will be correspondingly long. In such situations, it may not be possible to use the matched-filter scheme described in this paper, since the length of matched filters currently used is limited by both cost and technology. On the other hand, for several applications (such as packet radio), serial search schemes using a correlator may require too much transmission overhead for the acquisition preamble. It is of interest, therefore, to devise alternative schemes [7] for acquisition and attempt to accurately evaluate the capacity of such schemes. In many applications of interest, tight control of network timing is possible and the initial timing uncertainty can be reduced. This should be exploited to the greatest degree possible by the acquisition scheme to provide good performance in the presence of multiple-access interference. Effects of the periodicity of the signature sequences used in practice must also be accounted for, since the acquisition process may be severely impaired by the periodic recurrence of bad cross correlation peaks. Further analytical results, together with extensive computer evaluations or simulations with deterministic sequences such as Gold sequences and maximal-length sequences [13], are required for evaluating the capacity of practical systems.

VI. Appendix

We first consider a synchronous system, and give some details of the asymptotic evaluation of the lower bound (9) on the false alarm probability. This is followed by a brief derivation of upper and lower bounds on capacity for an asynchronous system. Finally, we provide a proof of Proposition 2. Throughout the Appendix, the notation $\underline{x}$ refers to a row vector, and $\underline{x}^T$ to its transpose.

A. Asymptotic Evaluation of the Lower Bound (9)

We are concerned with obtaining asymptotic values for terms of the form $P[Y_m/N > \alpha,\ Y_n/N > \alpha]$ for $m \neq n$ in a synchronous system; these terms appear in the lower bound (9) on the false alarm probability. Since (Y_n) is a stationary sequence, we can, without loss of generality, let $m = 0$ and $n \geq 1$. For any j,

$$Y_j = \sum_{i=0}^{N-1} h_i X_{j-i}. \tag{A.1}$$

For a synchronous system, the X_{j-i} are independent and identically distributed symmetric binomial random variables with parameter J, and are independent of the h_i, which are independent and identically distributed symmetric Bernoulli random variables. Using this, we can write

$$(NJ)^{-1/2}(Y_0, Y_n) = N^{-1/2}\{\sum_{i=0}^{N-n-1} \underline{A_i} + \sum_{i=N-n}^{N-1} \underline{B_i}\},$$

where

$$\underline{A_i} = J^{-1/2}(h_i X_{-i}, h_{i+n} X_{-i}),$$

and

$$\underline{B_i} = J^{-1/2}(h_i X_{-i}, h_{i+n-N} X_{N-i}).$$

The random vectors $\underline{A_i}$ and $\underline{B_i}$ are all independent. The $\underline{A_i}$ are identically distributed and are said to be of type 1, and the $\underline{B_i}$ are identically distributed and are said to be of type 2. Type 1 and type 2 random vectors both have mean zero and covariance matrix I_2, where I_s denotes the $s \times s$ identity matrix. However, their distributions are different. Specifically, if U, V and W are independent and identically distributed symmetric binomial random variables with parameter J, and h is a symmetric Bernoulli random variable, then $J^{-1/2}(U, hU)$ is of type 1, and $J^{-1/2}(V, W)$ is of type 2.

We now need a large deviations version of the central limit theorem for random vectors. Detailed discussion of the result is beyond the scope of this paper, hence we state the result without giving all the conditions required for it to hold. The pertinent references are [1] and [14]. Let $\underline{S_n}$ denote the sum of n independent random vectors in R^s with mean zero and covariance matrix I_s. Let D be a convex Borel set in R^s with $\underline{a}$ the point in the closure of D that is closest to the origin. Let μ denote the probability measure on R^s induced by a standard Gaussian random vector (mean zero, covariance matrix I_s); that is, if $\underline{X}$ is such a vector, then for any Borel set A, we have by definition that $\mu(A) = P[\underline{X} \in A]$. Then, given that $||\underline{a}|| = o(n^{1/6})$, and given certain additional conditions on D and the random vectors involved, we have the following multidimensional large deviations version of the central limit theorem:

$$P\left[n^{-1/2}\underline{S_n} \in D\right] = \mu(D)(1+o(1)), \qquad n \to \infty.$$

The details of checking that this result indeed applies in the situation of interest here are omitted. Since the random vector $(NJ)^{-1/2}$ (Y_0, Y_n) is a normalized sum of N independent random vectors of either type 1 or type 2, we conclude that as $N \to \infty$, it converges to a two-dimensional Gaussian random vector with mean zero and covariance matrix I_2, and, if $N/J^{3/2} \to 0$, the large deviations probability of interest is evaluated as

$$\begin{aligned} &P[Y_0/N > \alpha, Y_n/N > \alpha] \\ &= P\left[Y_0/(NJ)^{1/2} > \alpha(N/J)^{1/2}, Y_n/(NJ)^{1/2} > \alpha(N/J)^{1/2}\right] \\ &= Q^2\left[\alpha(N/J)^{1/2}\right](1+o(1)), \end{aligned}$$

which is the required result.

B. Bounds on Capacity for an Asynchronous Multiple-Access System

Upper and lower bounds on capacity that differ by a factor of four are derived for the asynchronous system described at the end of Section III. We *condition* on the θ_j and the τ_j unless explicitly stated otherwise. The input sequence X_k (due to the interference) to the matched filter is given by (1), where the h_i and the $x_k^{(j)}$ are independent and identically distributed symmetric Bernoulli random variables. Due to chip-asynchronism, the X_k are no longer independent, and some manipulations are needed for asymptotic evaluations of the upper and lower bounds on P_{FA} via the central limit theorem.

We first consider the term $P[Y_0/N > \alpha]$ in the upper bound (8). Define

$$V_k = \sum_{j=1}^{J} \sqrt{P_j} \cos\theta_j \, \tau_j x_{-k}^{(j)},$$

and

$$W_k = \sum_{j=1}^{J} \sqrt{P_j} \cos\theta_j \, (1-\tau_j) x_{-k}^{(j)}.$$

Note that (V_k, W_k) is an independent and identically distributed sequence of random vectors, although, for a given k, V_k and W_k are not independent. The matched filter input at time $-k$ is given by

$$X_{-k} = V_{k+1} + W_k. \tag{A.2}$$

Define

$$\sigma_0^2 = \sum_{j=1}^{J} P_j \cos^2\theta_j \left[\tau_j^2 + (1-\tau_j)^2\right], \tag{A.3}$$

and define the random variables

$$C_i = \sigma_0^{-1}(h_{i-1}V_i + h_i W_i), 1 \le i \le N-1, \tag{A.4}$$

$$D = \sigma_0^{-1}(h_{N-1}V_N + h_0 W_0). \tag{A.5}$$

From (A.1) and (A.2),

$$\sigma_0^{-1} Y_0 = D + \sum_{i=1}^{N-1} C_i. \tag{A.6}$$

The random variables C_i and D are independent, and have mean 0 and variance 1 (the C_i are identically distributed). It is easy to see that D is independent of the C_i, since V_N and W_0 in (A.5) are independent, symmetric random variables that do not appear in the expression (A.4) for the C_i for the range of i of interest. To show that the C_i are independent, it suffices to show that for any i, C_{i+1} is independent of $C_1, \cdots, C_i$. The random variable $C_i + 1$ depends on h_i,h_{i+1}, V_{i+1} and W_{i+1}; among the latter, only the symmetric Bernoulli random variable h_i is involved in the expressions for $C_1, \cdots, C_i$. Note that the random vectors (V_k, W_k) are independent, and that (V_k, W_k) has the same distribution as $(-V_k, -W_k)$. Further, $h_0, \cdots, h_{i-1}$ are independent and symmetric random variables. Using these facts, it is easy to see that knowing the values of $C_1, \cdots, C_i$ does not convey any information about the sign of h_i, and hence does not convey any information about C_{i+1}. This proves the independence of the C_i. At this point, it is appropriate to note that in the following, we will assert without proof the independence of other collections of random vectors; the proofs are based on arguments similar to the one used in this paragraph.

Normalizing each side of (A.6) by multiplying by $N^{-1/2}$, we obtain, using the large deviations version of the central limit theorem, that

$$P[Y_0/N > \alpha] = Q\left[\alpha N^{1/2}/\sigma_0\right](1+o(1)), \tag{A.7}$$

provided $N/\sigma_0^3 \to 0$.

The extreme right-hand side of (A.7) increases with σ_0, and it is easy to see from (A.3) that the maximum value of $\sigma_0 = (J\bar{P})^{1/2}$ is achieved if, for $1 \le j \le J$, we set $\theta_j = 0$ and $\tau_j = 0$. Using this value of σ_0, we proceed exactly as for the synchronous system to evaluate the upper bound on P_{FA}, and obtain the corresponding lower bound on capacity to be $\frac{1}{2}\alpha^2(N/\log N)/\bar{P}$. Note that the condition N/σ_0^3 for applying the large deviations result is satisfied if we consider the the maximum value of σ_0 for this value of J.

For convenience, we consider the worst-case value of $\tau = N-1$ for deriving the upper bound on capacity (our result depends only on τ being $O(N)$). In order to deal with the dependencies caused by chip-asynchronism, the lower bound on P_{FA} used to derive the upper bound on capacity is different from (9), and is given by

$$\begin{aligned} P_{\text{FA}} &\ge P\left[\cup_{r=0}^{M-1}\{Y_{\Delta r}/N > \alpha\}\right] \\ &\ge MP[Y_0/N > \alpha] \\ &\quad - \sum_{r=0}^{M-1}\sum_{s=r+1}^{M-1} P[Y_{\Delta r}/N > \alpha, Y_{\Delta s}/N > \alpha], \end{aligned} \tag{A.8}$$

where $M = \lfloor N/\Delta \rfloor$, and $\Delta > 1$ is an integer that tends to infinity as $N \to \infty$. By virtue of Proposition 2, it is only necessary to show that, for any $\delta > 0$, $P_{\rm FA}$ is bounded away from zero as $N \to \infty$ when

$$J = \{2\alpha^2 (N/\log N)/\overline{P}\}(1+\delta). \tag{A.9}$$

In order to do this, we must make an appropriate choice (specified later) of the parameter Δ in (A.8) in terms of both N and δ. It is worth noting that letting $\Delta \to \infty$ is merely a proof technique; for finite N, the lower bound (9) (which corresponds to $\Delta = 1$), together with an approximation regarding the independence of the Y_n, would probably yield a better approximate lower bound for the false alarm probability.

For an asymptotic evaluation of (A.8), we consider, using the stationarity of the Y_n as in Section A.1, the term $P[Y_0/N > \alpha, Y_n/N > \alpha]$, where $n \geq \Delta$. The matched filter output Y_0 can be expressed as in (A.6), and the corresponding expression for Y_n is

$$\sigma_0^{-1} Y_n = \sigma_0^{-1} \sum_{i=1}^{N-1} (h_{i-1} V_{i-n} + h_i W_{i-n}) + (h_{N-1} V_{N-n} + h_0 W_{-n}).$$

In order to apply the central limit theorem, we must express (Y_0, Y_n) as a sum of independent vectors. To this end, we consider first the situation in which $N - 1 - n \leq n$, and define the following random vectors in R^2:

$$\begin{aligned}\underline{E_i} = 2^{-1/2}\sigma_0^{-1}(&h_{i-1}V_i + h_i W_i + h_{i+n-1}V_{i+n} \\ &+ h_{i+n}W_{i+n}, h_{i-1}V_{i-n} + h_i W_{i-n} \\ &+ h_{i+n-1}V_i + h_{i+n}W_i), \\ 1 \leq i \leq N-1-n,&\end{aligned}$$

$$\underline{F_i} = \sigma_0^{-1}(h_{i-1}V_i + h_i W_i, h_{i-1}V_{i-n} + h_i W_{i-n}), \quad N - n \leq i \leq n,$$

and

$$\underline{G} = \sigma_0^{-1}(h_{N-1}V_N + h_0 W_0, h_{N-1}V_{N-n} + h_0 W_{-n}).$$

Then

$$\sigma_0^{-1}(Y_0, Y_n) = \underline{G} + 2^{1/2} \sum_{i=1}^{N-1-n} \underline{E_i} + \sum_{i=N-n}^{n} \underline{F_i}. \tag{A.10}$$

The contribution of the random vector $\underline{G}$ to the right-hand side above is asymptotically negligible, and is ignored henceforth. It is easy to see that the vectors $\underline{E_i}$ and $\underline{F_i}$ are independent, and each have mean zero and covariance I_2. If $N-1-n = o(N)$, the contribution of the $\underline{E_i}$ is asymptotically negligible, and the central limit theorem can be applied to the second summation in (A.10). Otherwise, the central limit theorem is applied separately to each of the two summations in (A.10), assuming $n = \beta N + o(N)$ for $1/2 \leq \beta < 1$. In either case, it is easy to conclude that

$$P[Y_0/N > \alpha, Y_n/N > \alpha] = Q^2\left[\alpha N^{1/2}/\sigma_0\right](1 + o(1)). \tag{A.11}$$

The situation when $n < N - 1 - n$ is more difficult to handle, and this is where the choice of Δ becomes critical. The appropriate summands for applying the central limit theorem in this case are the random vectors found in the equation at the bottom of the page. By virtue of the above definition, dependent terms that contribute to the random vector (Y_0, Y_n) are grouped into each of the $\underline{H_i}$ in a manner such that the $\underline{H_i}$ are independent. Each of the $\underline{H_i}$ has mean zero and asymptotic covariance I_2, and it can be shown that

$$\sigma_0^{-1} N^{-1/2}(Y_0, Y_n) = n^{-1/2} \sum_{i=1}^{n} \underline{H_i} + o(1).$$

Since $\Delta \to \infty$ and $n \geq \Delta$, the central limit theorem can now be applied to obtain the asymptotic evaluation (A.11). As before, we need a large deviations version of the central limit theorem, which requires the condition $N^{1/2}/\sigma_0 = o(n^{1/6})$. For this condition to hold, it is sufficient that

$$N^{1/2}/\sigma_0 = o\left(\Delta^{1/6}\right). \tag{A.12}$$

Substituting (A.7) and (A.11) in (A.8), we have the asymptotic lower bound $g(r_0)$ for $P_{\rm FA}$, where $g(x) = x - x^2/2$, and

$$r_0 = \lfloor N/\Delta \rfloor Q\left[\alpha N^{1/2}/\sigma_0\right].$$

A lower bound on σ_0 is obtained by setting $\tau_j = 1/2$ for all j in (A.3), so that

$$\sigma_0^2 \geq \tfrac{1}{2}\sum_{j=1}^{J} P_j \cos^2\theta_j = \tfrac{1}{4}\sum_{j=1}^{J} P_j + \tfrac{1}{4}\sum_{j=1}^{J} P_j \cos 2\theta_j.$$

If the θ_j are independent and uniform over $[0, 2\pi]$, the second term on the right-hand side above is a symmetric random variable, so that, with probability at least $1/2$,

$$\sigma_0^2 \geq \tfrac{1}{4}\sum_{j=1}^{J} P_j = J\overline{P}/4, \tag{A.13}$$

For $r \in [0,1]$, $g(r)$ is positive and increasing. Since r_0 increases with σ_0, on averaging over $\underline{\theta} = (\theta_1, \cdots, \theta_J)$, we obtain

$$E_{\underline{\theta}}\{P_{\rm FA}\} \geq \tfrac{1}{2} g(r_1),$$

where $r_1 = \lfloor N/\Delta \rfloor \, Q[\alpha(4N/J\overline{P})^{1/2}]$. For J as in (A.9), we obtain, using the asymptotic evaluation for $Q(\cdot)$, that

$$r_1 = 2^{-1/2}\Delta^{-1} N^{\delta/1+\delta} (\log N)^{-1/2}(1 + o(1)).$$

$$\underline{H_i} = \sigma_0^{-1}(N/n)^{-1/2} \sum_{j=1}^{\lfloor (N-1)/n \rfloor - 1} (h_{i+jn-1}V_{i+jn} + h_{i+jn}W_{i+jn}, h_{i+jn-1}V_{i+jn-n} + h_{i+jn}W_{i+jn-n}), 1 \leq i \leq n.$$

We show that $E_{\theta}\ \{P_{\text{FA}}\}$ is bounded away from zero by showing that r_1 is bounded away from zero, and the latter is achieved for

$$\Delta = O\Big(N^{\delta/1+\delta}(\log N)^{-1/2}\Big). \tag{A.14}$$

It is easy to check that, for J as in (A.9) and σ_0 as in (A.13), Δ can be chosen to simultaneously satisfy the conditions (A.12) and (A.14). This completes the derivation of the upper bound on capacity.

C. Proof of Proposition 2

We condition throughout on the filter coefficients. Denote by Y_n the matched filter output at time n due to the J original interfering transmissons, Z_n the output due to the additional interference, and Y_n' the net output due to the interference. The sequences $\{Y_n\}$ and $\{Z_n\}$ are independent, and since the filter is linear, we have $Y_n' = Y_n + Z_n$. Note that since the interference input is symmetric, the Z_n are symmetric random variables. Let $M = \min\{0 \le n \le \tau - 1 | Y_n = \max_{0\le i \le \tau-1} Y_i\}$. Then

$$\begin{aligned} P_{\text{FA}}' &= P\left[\max_{0\le n\le \tau-1}(Y_n + Z_n) > \alpha N\right] \\ &\ge P[Y_M > \alpha N, Z_M \ge 0] = P[Y_M > \alpha N]P[Z_M \ge 0] \\ &= \tfrac{1}{2}P_{\text{FA}}, \end{aligned}$$

where we have used the facts that the Y_n and M are independent of the Z_n, that the Z_n are symmetric, and that $P_{\text{FA}} = P[Y_M > \alpha N]$. This completes the proof. □

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