ON THE LIMITS OF COMMUNICATION PERFORMANCE WITH ONE-BIT ANALOG-TO-DIGITAL CONVERSION

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ABSTRACT

As communication systems scale up in speed and bandwidth, the power consumption and cost of high-precision (e.g., 6-12 bits) analog-to-digital conversion (ADC) becomes the limiting factor in modern receiver architectures based on digital signal processing. In this paper, we consider the effects of lowering the precision of the ADC on the performance of the communication link. We focus on the most extreme scenario, in which the receiver employs one-bit ADC at baseband. While this constraint impacts every aspect of design ranging from synchronization and channel estimation to demodulation and decoding, in this initial exposition, we focus on the Shannon-theoretic limits imposed by one-bit ADC for an ideal discrete-time real baseband Additive White Gaussian Noise (AWGN) channel. Results are obtained both for non-spread and spread spectrum systems. While binary phase shift keying (BPSK) is optimal in a non-spread system, we obtain less predictable results for spread spectrum system, including the sub-optimality of BPSK and non-monotonicity of mutual information with signal-to-noise ratio for a fixed constellation.

1. INTRODUCTION

Digital signal processing (DSP) forms the core of modern digital communication receiver implementations, with the analog baseband signal being converted to digital form using ADCs with typically 6-12 bits of precision However, as we scale the bandwidths and speeds of communication systems, the cost and power consumption of the ADC required to maintain such precision becomes prohibitive[1]. One possible approach in this scenario is to live with lower precision ADC. Such a design choice impacts all aspects of receiver design, such as carrier and timing synchronization, equalization, and the generation of soft information for decoding. However, before embarking on a comprehensive rethinking of communication system design, it is important to understand the fundamental limits on communication performance imposed by low-precision ADC. In this paper, we take an initial step in this direction, investigating the Shannon-theoretic limits imposed by the use of one-bit ADC for linear modulation over a real baseband AWGN channel, with ideal sampling of the received signal at the symbol rate. In addition to standard linear modulation, we also consider direct sequence (DS) spread spectrum, for which low-cost implementations often implement one bit ADC.

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Our main result for standard linear modulation is that, for symbol rate Nyquist sampling with one bit ADC, the use of BPSK is optimal. Of course, once we commit to using BPSK, the degradation due to the use of hard decisions (i.e., one bit ADC) relative to soft decisions (i.e., no limit on ADC precision) is the well-known figure of 1.97 dB [3]. It is worth noting that, for discrete memoryless channels, Gallager's classic text on information theory [2] shows that the number of "active" inputs need not exceed the number of outputs. We cannot directly apply Gallager's results because we consider a channel with power constraints, and employ a direct proof, but leave open the question of whether Gallager's proof can be adapted to our scenario.

For DS spread spectrum systems, on the other hand, we show that BPSK, which is often the modulation of choice in such systems, is actually suboptimal even when the receiver employs one bit ADC on its chip-rate samples. Intuitively, this is because the despread output exhibits more than two levels, and can therefore be used to distinguish between a larger number of input levels. We demonstrate that a ternary alphabet $\{0, \pm 1\}$ outperforms BPSK, while more sophisticated constellations provide even greater gains. The gains over BPSK are most significant at larger processing gains and/or larger 'chip SNR' (henceforth SNR). Another interesting feature of spread spectrum systems is that, for a fixed constellation, the mutual information is actually not monotonic in SNR. While we do not find the capacity and optimal constellation for spreading gain greater than 1, we use a heuristic based on the uncoded case and the plane-cutting algorithm [5] to construct good constellations.

The next section describes the results for unspread systems. Spread spectrum systems are considered in Section 3, followed by the conclusions in Section 4.

2. BPSK IS OPTIMAL

We assume that the received signal satisfies the Nyquist criterion [3, pp. 543]. Hence the discrete time channel obtained after sampling is memoryless and in case of one-bit ADC it is given by:

$$Y = \text{sign}(X + \sigma W). \tag{1}$$

Here X is the input satisfying the power constraint $\mathsf{E}[X^2] \leq P$, W is $\mathcal{N}(0,1)$ and is independent of X. Our first result identifies the capacity of this channel.

Theorem 1. *The capacity of channel* (1) *is achieved by BPSK signalling and is given by*

$$C = 1 - h\left(Q\left(\sqrt{\mathsf{SNR}}\right)\right), \ \mathsf{SNR} = \frac{P}{\sigma^2}$$

where

$$h(a) = -a\log(a) - (1-a)\log(1-a)$$

and $Q(x)$ is the complementary Gaussian distribution function:

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-t^2/2)dt.$$

Proof. The capacity

$$C = \max_{\mathsf{E}[X^2]\leq P} I(X,Y).$$

Since Y is binary it is easy to see that

$$H(Y|X) = \mathsf{E}\left[h\left(Q\left(\frac{X}{\sigma}\right)\right)\right].$$

Therefore

$$I(X,Y) = H(Y) - \mathsf{E}\left[h\left(Q\left(\frac{X}{\sigma}\right)\right)\right]$$

which we wish to maximize over all input distributions satisfying $\mathsf{E}[X^2] \leq P$. We note that if X is symmetric random variable (that is X and $-X$ have the same distribution), then

$$P(Y=1) = P(Y=-1) = 0.5$$

and $H(Y)$ takes its maximum value of 1. Now given a distribution function $F(x)$, we can construct another distribution function

$$G(x) = \frac{F(x)}{2} + \frac{1-F(-x)}{2}$$

such that

- $G(x)$ is a mixture of the distribution $F(x)$ of X and the distribution $1 - F(-x)$ of $-X$.
- $G(x)$ is the distribution of a symmetric random variable and hence for this distribution on X, $H(Y)$ attains its maximum value 1.
- $h(Q(x/\sigma))$ is a symmetric function and hence $H(Y|X)$ is the same whether X is distributed as per $F(x)$ or $G(x)$.

As a consequence of these facts, we can restrict our maximization only to symmetric X and we get

$$C = 1 - \min_{\substack{X \text{ symmetric} \\ \mathsf{E}[X^2]\leq P}} \mathsf{E}\left[h\left(Q\left(\frac{X}{\sigma}\right)\right)\right].$$

Since $h(Q(y))$ is an even function, we get that

$$H(Y|X) = \mathsf{E}\left[h\left(Q\left(\frac{X}{\sigma}\right)\right)\right] = \mathsf{E}\left[h\left(Q\left(\frac{|X|}{\sigma}\right)\right)\right].$$

Below we show that $h(Q(\sqrt{y}))$ is convex in y and hence invoking Jensen's inequality [4], we get

$$H(Y|X) \geq h\left(Q\left(\sqrt{\mathsf{SNR}}\right)\right)$$

with equality iff $X^2 = P$. Coupled with the symmetry condition on X, this implies that BPSK achieves capacity and the capacity is

$$C = 1 - h\left(Q\left(\sqrt{\mathsf{SNR}}\right)\right).$$

To complete the proof, we need to establish the convexity of $h(Q(\sqrt{y}))$. One way to prove convexity of $h(Q(\sqrt{y}))$ is to show that the second derivative is positive everywhere. It is tedious but straightforward to show this for y sufficiently large; for other values the positivity is demonstrated in Figure 1. □

3. DIRECT SEQUENCE SPREAD SPECTRUM

Direct sequence spread spectrum communication is commonly used in practice in the low spectral efficiency regime. In this section we consider the case when the spreading gain is $K > 1$, that is, each symbol is transmitted using K chips. We assume that the Nyquist criterion is satisfied at the chip level and hence after sampling there is no inter-chip interference. In this case, the K output samples corresponding to input symbol X are

$$Y_k = \text{sign}(c_k X + \sigma W_k), \quad 1 \leq k \leq K$$

where $\{W_k\}$ are i.i.d. $\mathcal{N}(0,1)$, $c_k \in \{1,-1\}$ is the chip sequence known to the transmitter as well as the receiver. By multiplying both sides by c_k it is easy to see that the channel is equivalent to

$$Z_k = \text{sign}(X + \sigma V_k), \quad 1 \leq k \leq K$$

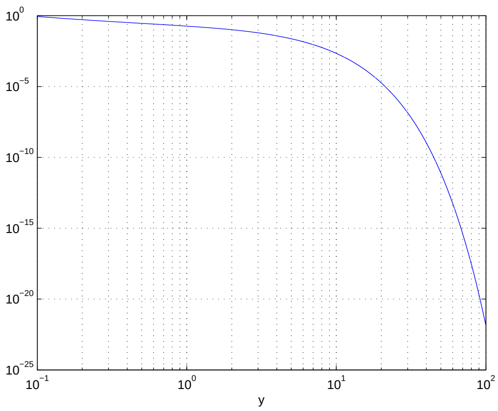

Fig. 1. The second derivative of $h(Q(\sqrt{y}))$ is positive everywhere.

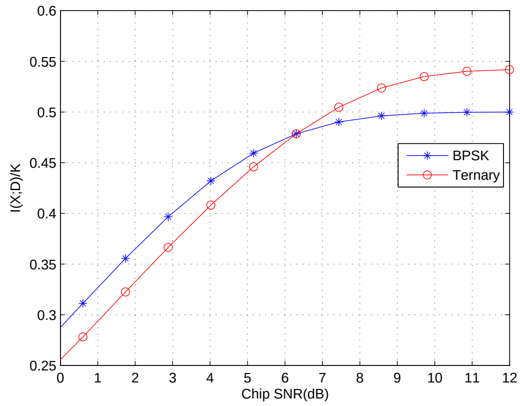

Fig. 2. Mutual information per chip ($K = 2$) for BPSK and ternary alphabets, showing that BPSK is not optimal.

where $\{V_k\}$ are i.i.d. $\mathcal{N}(0, 1)$, or in vector notation,

$$\boldsymbol{Z} = \text{sign}(X + \sigma \boldsymbol{V})$$

where $\boldsymbol{V}$ is $\mathcal{N}(0, I)$. In the remainder of this section, we study the sufficient statistic for this channel, the input-output mutual information, and construct good constellations to transmit over this channel.

3.1. Sufficient Statistic

Let D := the number of ones in $\boldsymbol{Z}$. We next prove that X and $\boldsymbol{Z}$ are independent conditioned upon D. Since V_k are i.i.d.,

$$\mathsf{P}(\boldsymbol{Z} = \boldsymbol{z} | X = x, D = d) = \frac{1}{\binom{K}{d}}.$$

Therefore,

$$\begin{aligned} \mathsf{P}(\boldsymbol{Z} = \boldsymbol{z} | D = d) &= \int_{\mathbb{R}} \mathsf{P}(\boldsymbol{Z} = \boldsymbol{z} | X = x, D = d) dF_X(x) \\ &= \frac{1}{\binom{K}{d}} \int_{\mathbb{R}} dF_X(x) \\ &= \mathsf{P}(\boldsymbol{Z} = \boldsymbol{z} | X = x, D = d). \end{aligned}$$

Thus X and $\boldsymbol{Z}$ are independent conditioned upon D, that is, $(X, D, \boldsymbol{Z})$ is a Markov chain. But we know by the data processing inequality [4] that

$$I(X; \boldsymbol{Z}) \geq I(X; D)$$

with equality if and only if $(X, D, \boldsymbol{Z})$ is a Markov chain. Thus we see that $I(X; \boldsymbol{Z}) = I(X; D)$ and we have shown the following proposition for direct sequence spread spectrum with 1-bit ADC.

Proposition 1. *D is a sufficient statistic and no information loss is encountered if the receiver just counts D.*

We note that

$$D = d \text{ iff } \sum_k c_k Y_k = 2d - K, \quad 0 \leq d \leq K.$$

Hence the usual despreader leads to no information loss in the 1-bit ADC case. This despreader is also simple to implement.

3.2. Non-optimality of BPSK for $K \geq 2$

The capacity with spreading gain K is given by

$$C_K = \max_{\mathsf{E}[X^2] \leq P} \frac{I(X; \boldsymbol{Z})}{K} = \max_{\mathsf{E}[X^2] \leq P} \frac{I(X; D)}{K}.$$

We note that even though for $K = 1$ BPSK is optimal, this is not necessarily the case for $K \geq 2$. For example, consider $K = 2$ and the ternary alphabet $[-1, 0, 1]$. When $\mathsf{P}(X = 0) = 0.15$, $\mathsf{P}(X = 1) = \mathsf{P}(X = -1) = 0.425$, it is seen in Figure 2 that for SNR > 6 dB the ternary input leads to a larger input-output mutual information than BPSK.

3.3. Non-monotonicity of Mutual Information with SNR

Another interesting phenomenon that we have observed is that for $K \geq 2$, depending upon input distribution, as SNR increases, the mutual information $I(X; Y)$ may not increase monotonically. For example, consider the case when X is uniformly distributed on $[-2, -1, 0, 1, 2]$ and $K = 8$. Figure 3 shows that as SNR increases, $I(X; Y)$ first reaches a global maximum, then a local minimum, and finally settles to a limiting value. This behavior has several practical implications. For example, if for the above 5-PAM constellation, the SNR is more than 7 dB, then the capacity can be increased by adding Gaussian noise to the received signal before quantization so that the SNR becomes 7 dB. In general, identifying the SNR

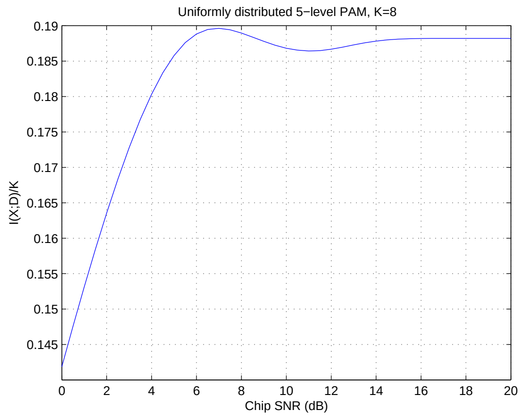

Fig. 3. Mutual information per chip ($K = 8$) for a uniformly distributed 5-PAM constellation is not monotonic with SNR.

at which the maximum occurs for a given input X seems analytically intractable, though the limiting value as $\sigma \to 0$ is easily found and it is given below.

Proposition 2. *Suppose* $\mathsf{P}(X > 0) = \mathsf{P}(X < 0) = a$ *and* $\mathsf{P}(X = 0) = 1 - 2a$. *Then*

$$\lim_{\sigma \to 0} I(X; D) = -\sum_{d=0}^{K} p(d) \log(p(d)) + (1 - 2a) \sum_{d=0}^{K} u(d) \log(u(d))$$

where

$$u(d) := \frac{1}{2^K} \cdot \binom{K}{d}$$

and

$$p(d) = a + \frac{1 - 2a}{2^K} \binom{K}{d}, \; d \in \{0, K\}$$
$$= \frac{1 - 2a}{2^K} \binom{K}{d}, \; d \in \{1, 2, \ldots, K - 1\}.$$

The proof is given in the Appendix. Note that two different constellations with an identical a have the same limiting value of mutual information, irrespective of what their actual transmit levels are.
The value of a that maximizes this limiting mutual information, for a given K, is found to be

$$\hat{a} = \frac{R}{2R + 2^K} \quad , \quad R = \exp\left(\frac{2^K K \ln(2)}{2^K - 2}\right) - 1$$

3.4. Construction of Good Constellation

Finding the capacity for $K \geq 2$ is a non-trivial task and we resort to the plane-cutting algorithm [5] to find it numerically. To ensure that the linear program employed by the algorithm is finite dimensional, we take the state space of the input to be finite. Next, we develop a heuristic for constructing good input constellations using the uncoded case.

In practice usually K is large. Now if the symbol x is transmitted, then for large K, by the central limit theorem, $(D - K * p)/\sqrt{K}$ is Gaussian with zero mean and variance $p(1 - p)$ where $p = Q(-x/\sigma)$. Note that to choose the constellation, we could choose the corresponding p-values. The p-values can be chosen so that the Gaussians centered at the respective p-values have small overlap. Thus we can construct symmetric constellations by choosing p-values symmetric around 0.5. We choose the first point to be $p_1 = 0.5$. When the corresponding symbol $x_1 = 0$ is transmitted, D/K is roughly Gaussian with mean 0.5 and standard deviation $\sqrt{\frac{p_1(1-p_1)}{K}} = 0.5/\sqrt{K}$. The next point $p_2 > p_1$ is chosen such that the $\alpha\sqrt{\frac{p_2(1-p_2)}{K}}$ point of the Gaussian centered at p_2 coincides with the $\alpha\sqrt{\frac{p_1(1-p_1)}{K}}$ point of the gaussian centered at p_1. Thus we are choosing the p-values such that the intersections of the Gaussians are at $\alpha\sqrt{\frac{p(1-p)}{K}}$ points. The constellation points can be obtained by inverting the Q-function and scaling to satisfy power constraint. For large α, we expect the hard demodulator to give small demodulation error and hence we expect this to be a reasonable choice of constellation.

Simulation Results: For $K = 32$ and $\alpha = 2$, the p-values obtained were $\{0.011, 0.186, 0.5, 0.814, 0.989\}$, and the corresponding constellation points were such that $\frac{x}{\sigma} = \{-2.287, -0.894, 0, 0.894, 2.287\}$. The power constraint P on the input was assumed to be 1, i.e., $\mathsf{E}[X^2] \leq 1$. Figure 4 compares the performance of the heuristic based constellation with that of BPSK, Ternary PAM ([-1 0 1]), and a 21 point PAM uniformly spread over the interval $[-2, 2]$.

In the low SNR regime, noise dominates over the signal and therefore BPSK leads to only small information loss [5]. However, as the SNR is increased, BPSK becomes quite suboptimal due to the absence of $\{x = 0\}$ transmit level, which can potentially induce enough randomness in the output so as to increase the mutual information. The limiting value for BPSK as $\sigma \to 0$ is $\frac{1}{K} = 0.0313$, which is expected since $H(X) \to 1$, while $H(X|Y) \to 0$. The ternary alphabet also performs worse than the heuristic based and 21 PAM constellations. However, its limiting value as $\sigma \to 0$ is identical to that for 21 PAM, since the limiting value depends only on the probabilities $\{\mathsf{P}(x > 0), \mathsf{P}(x = 0), \mathsf{P}(x < 0)\}$, and not on the choice of alphabet that achieves them. Note that the heuristic constellation does not show non-monotonicity in mutual information since its choice is governed by σ. For

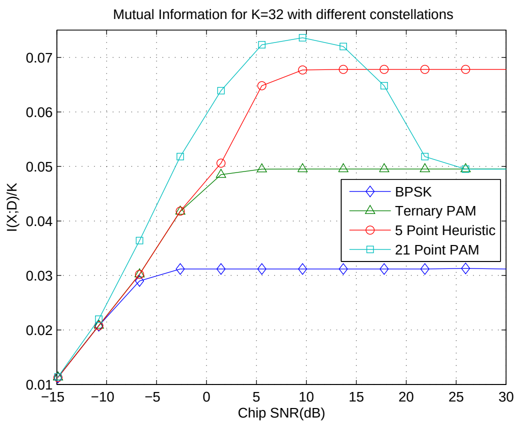

Fig. 4. Mutual information per chip ($K = 32$), comparing a 5 point constellation optimized based on CLT heuristics with BPSK, ternary signaling, and 21-PAM.

high SNR($\sigma \to 0$), the transmit levels in the heuristic based constellation are also concentrated around 0, which is not the case with a fixed constellation.

4. CONCLUSIONS

The results presented here are but a first step towards understanding the fundamental limits of communication performance as a function of the speed and precision of receiver ADC. We have seen that one-bit modulation (BPSK) is optimal for one-bit ADC with symbol rate sampling. The results for fractionally spaced sampling are expected to be different, and to bring up interesting tradeoffs regarding the speed versus the precision of ADC. Analogous results for multiple (e.g., 2-4) bits of ADC precision are currently being investigated.

The suboptimality of BPSK for DS spread spectrum systems with one-bit ADC is not really that surprising in retrospect. Even though the *input* to the despreader has a precision of one bit per chip, the precision of the *output* of the despreader is larger than one, and can therefore convey more than one bit of information per spread symbol. The high SNR regime in which we see the nonmonotonicity of mutual information (for a given constellation) may not be of practical significance, since spread spectrum is usually employed at low SNRs. However, we see that even at moderate SNRs, there are significant gains to be obtained from the use of non-BPSK alphabets. For example, for a processing gain of 32, we can see significant gains from constellation optimization even at a SNR of 0 dB.

5. APPENDIX

Proof of Proposition 2: To find the limiting value of $I(X;D)$, we first note that as $\sigma \to 0$, the conditional probability mass function (PMF) $\mathsf{P}(D = d|X = x)$ tends to the unit mass at $d = K$ for $x > 0$, $\mathsf{P}(D = d|X = x)$ tends to the unit mass at $d = 0$ for $x < 0$, and

$$\mathsf{P}(D = d|X = 0) \to \frac{1}{2^K} \cdot \binom{K}{d} = u(d). \qquad (2)$$

The last limit follows from the fact that the Gaussian density is symmetric and therefore when $x = 0$ is transmitted all 2^K output sequences are equally likely. Since $\mathsf{P}(X > 0) = \mathsf{P}(X < 0) = a$ and $\mathsf{P}(X = 0) = 1 - 2a$, the limit of $\mathsf{P}(D = d|X = x)$ is the mixture $p(d)$.

Using the fact that the entropy of a finite alphabet random variable is a continuous function of its probability law, we get

$$\begin{aligned}\lim_{\sigma\to 0} H(D) &= H(\lim_{\sigma\to 0} \mathsf{P}(D)) \\ &= -\sum_{d=0}^{K} p(d)\log(p(d)).\end{aligned}$$

Now

$$H(D|X) = \int_{\mathbb{R}} H(D|X = x) F_X(dx).$$

Since $H(D|X = x) \leq \log(K + 1)$, by the dominated convergence theorem,

$$\begin{aligned}\lim_{\sigma\to 0} H(D|X) &= \int_{\mathbb{R}} \lim_{\sigma\to 0} H(D|X = x) F_X(dx) \\ &= -(1 - 2a)\sum_{d=0}^{K} u(d)\log(u(d))\end{aligned}$$

where $u(d)$ is as defined in (2) above.

The limiting value of mutual information is now obtained as

$$\lim_{\sigma\to 0} I(X;D) = \lim_{\sigma\to 0} H(D) - \lim_{\sigma\to 0} H(D|X).$$

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