Limiting Performance of Frequency-Hop Random Access

UPAMANYU MADHOW AND MICHAEL B. PURSLEY, FELLOW, IEEE

Abstract —The multiple-access capability of asynchronous frequency-hop packet radio networks is analyzed. The only interference considered is multiple-access interference, and perfect side information is assumed. Bounds are developed on the probability of error for unslotted systems based on the distributions of the maximum and minimum interference levels over the duration of a given packet, and these are employed to develop corresponding bounds on the throughput. Our idealized model enables the derivation of asymptotic results showing the convergence of these bounds for high traffic levels, and the asymptotic performance of the system is seen to be the same as that of the corresponding slotted system. Results are also obtained for the maximum asymptotic throughput. These results shows that the asymptotic sum capacity of the channel can be attained using Reed–Solomon coding. All these results are valid for either fixed or exponentially distributed packet lengths. Our results indicate that the performance of frequency-hop networks is insensitive both to the distribution of packet lengths and to whether or not transmissions are slotted. It also demonstrates the efficacy of Reed–Solomon coding in combating multiple-access interference.

I. Introduction

We analyze the effect of multiple-access interference on the throughput of fully connected asynchronous frequency-hop (FH) packet radio networks. To obtain meaningful analytical results, we ignore all other forms of interference and assume that perfect side information is available to the receiver. The network population is infinite, and the traffic is assumed to be Poisson. Because the side information is perfect, the FH channel is an erasures-only channel, and bounded-distance erasures-only decoding is used. Both our model and our general approach for analyzing unslotted systems are based on [11]. We develop upper and lower bounds on the packet error probability for unslotted systems, and we show that these bounds converge at high traffic levels to the asymptotic error probability for the slotted system derived in [11]. It is also shown that the asymptotic maximum possible throughput of the channel using Reed–Solomon (RS) coding is equal to the asymptotic sum capacity of the channel [7].

Before proceeding further, we clarify our terminology and summarize some existing results. The terms slotted and unslotted, as applied to a network, simply specify the access protocol to be slotted or unslotted Aloha, respectively. The frequency hopping is said to be synchronous if the dwell intervals from different terminals are aligned at each receiver, and it is *asynchronous* if they are not. If the number of active terminals in a slotted network is *fixed*, it can be viewed as a fixed discrete memoryless multiple-access erasures-only channel, and it has been shown in [7] that the asymptotic sum capacity of the channel is e^{-1} for synchronous hopping and $e^{-1}/2$ for asynchronous hopping. Using the same network model, it is shown in [8] that the asymptotic maximum possible throughput using Reed–Solomon coding is equal to the asymptotic sum capacity.

In contrast with the previous work, we focus on the coded performance of a system that has a variable traffic level, and we consider both fixed- and variable-length packets. Since the asymptotic performance of slotted and unslotted networks is the same, it seems that the performance of FH systems in general would be insensitive to whether there is slotting in the network. Note, however, that while the use of frequency hopping removes sensitivity to alignment at the packet level, an alignment of the dwell intervals (i.e., synchronous hopping) results in a twofold improvement in the throughput performance. Our results are valid for packet lengths that are exponentially distributed as well as for packets of fixed length, and they strongly suggest that the distribution of packet lengths is not a significant factor in the performance of an FH packet radio network.

A general description of the model and the method of analysis is presented in Section II. The distributions for the maximum and minimum interference levels are used to bound the error probability. The result for fixed-length packets derived in [11] is given at the end of Section II. The corresponding distributions for variable-length packets are derived in Section III. The asymptotic results of Section IV have the same form for both slotted and unslotted systems, as well as for fixed and exponentially distributed packet lengths, which enables us to present them in a general framework. Our conclusions are given in Section V.

II. Model and Analysis

Consider a fully connected network with an infinite population of identical terminals. Each terminal can transmit and receive at the same time, and there is no queuing of packets at the terminals. Also, each terminal

Manuscript received January 4, 1988; revised June 15, 1989. This work was supported by the Army Research Office under Contract DAAL-03-87-K-0097. This work was presented in part at the 1988 IEEE International Symposium on Information Theory, Kobe, Japan, June 19–24.

The authors are with the Coordinated Science Laboratory, University of Illinois, 1101 West Springfield Avenue, Urbana, IL 61801.

IEEE Log Number 8934000.

0018-9448/90/0300-0322$01.00

can receive any number of packets simultaneously. As far as the network model is concerned, these assumptions are made without any loss of generality, since the network population is infinite. We assume a Poisson packet generation process with an average rate of λ packets per second. This is the net arrival rate of packets in the network, including retransmissions, if any. The network is assumed to be in steady state for the purpose of our analysis, and stability issues are not considered. Since our purpose is to evaluate the effects of multiple-access interference and determine the maximum multiple-access capability, thermal noise and other types of interference are not considered.

The radios employ frequency hopping with one transmitted symbol per dwell interval over a band of q frequency slots. The hopping patterns are random, the hopping patterns for different ratios are independent, and the hopping patterns are independent of the transmitted data. If two or more terminals are transmitting at the same time, a *hit* is said to occur when the two different signals simultaneously occupy the same frequency slot. Consider a single symbol from one of the packets. The probability that the other signal occupies the same frequency slot at any time during the interval occupied by this symbol is called the probability of a hit and is denoted by P_h. For most cases of interest, $P_h = cq^{-1} + O(q^{-2})$ for a suitable choice of the constant c [6], [11]. It is assumed that all hits are detected (perfect side information), and that all symbols involved in a hit are erased by the receiver. An (n,k) block code that can correct up to e erasures is employed with bounded-distance erasures-only decoding, so the codeword is in error when there are more than e erasures.

A packet may consist of more than one codeword, and the codeword errors may or may not be dependent, so the relation between the packet and codeword error probabilities may be quite complicated [11]. For much of the development in this paper, therefore, we restrict our attention to the codeword error probability. Another performance measure of interest is the normalized throughput, which is defined in Section IV. In deriving the results for the normalized throughput, it does become necessary to consider the relation between the packet error probability and the codeword error probability. For the purpose of deriving our asymptotic results, however, it suffices to work with fairly loose estimates of the packet error probability that do not require a detailed knowledge of the dependence of the codeword errors.

For our analysis we consider a given packet, called the tagged packet, which contains the codeword of interest. The probability of error for this codeword depends on $N(t)$, the number of interfering transmissions in the network during the interval $[0,T]$ in which the tagged packet is transmitted. Consider first a slotted system. For a synchronous frequency hopping model, the hits are conditionally independent given the number of interfering packets, and $c = 1$ in the expression for P_h given earlier. For an asynchronous model, the hits are not conditionally independent [5], but for our purposes, we can assume conditional independence and use $c = 2$ in the expression for P_h. As shown in [5], this provides an upper bound on the actual codeword error probability, and for high traffic levels, the hits do become conditionally independent. The probability that a given symbol is hit, and hence erased, is $p_j = 1-(1-P_h)^j$, if $N(t) = j$ for $0 < t < T$, and the conditional probability of codeword error is

$$P_E(j) = \sum_{i=e+1}^{n} \binom{n}{i} p_j^i (1-p_j)^{n-i}. \qquad (2.1)$$

Let $E\{T\}$ be the mean length of the packets. For fixed-length packets, $E\{T\} = T$. For variable-length packets, we assume T has an exponential distribution with parameter μ, so $E\{T\} = \mu^{-1}$. The net arrival rate of packets in the network is λ packets per second. The normalized traffic level, or offered traffic, is $\rho = \lambda E\{T\}$, so $\rho = \lambda T$ for fixed-length packets, and $\rho = \lambda/\mu$ for variable-length packets. Let f_ρ be the probability mass function and F_ρ the distribution function for a Poisson random variable with mean ρ. Then the probability of error for a slotted system is given by [11]

$$P_E^S = \sum_{j=0}^{\infty} f_\rho(j) P_E(j).$$

For unslotted systems, $N(t)$ need not be constant over the duration of the tagged packet, so the average probability of error is difficult to compute. Define $N^* = \max\{N(t): t \in [0,T]\}$ and $N_* = \min\{N(t): t \in [0,T]\}$, which are the maximum and minimum interference levels during the transmission of the tagged packet. Let f^* and f_* be the probability mass functions corresponding to the distributions of N^* and N_*, respectively, and let F^* and F_* be the corresponding distribution functions. The following observations are made in [11]:

$$P(E|N^* = j) \le P_E(j) \le P(E|N_* = j),$$

$$P_E = \sum_j P(E|N^* = j) f^*(j) = \sum_j P(E|N_* = j) f_*(j),$$

and

$$P_E^L = \sum_j P_E(j) f_*(j) \le P_E \le \sum_j P_E(j) f^*(j) = P_E^U.$$

The distributions of N^* and N_* are derived in [11] for fixed-length packets, and we have the following expressions for the corresponding probability mass functions.

$$f^*(j) = \frac{f_\rho(j)}{j+1}\Big\{\rho(j+1-\rho) f_\rho(j) + \big[(j+1-\rho)^2 + \rho\big] F_\rho(j)\Big\}$$

and

$$f_*(j) = f_\rho(j)\left\{(\rho+1-j) f_\rho(j) + \frac{1}{\rho}\big[(\rho-j)^2+\rho\big]\big[1-F_\rho(j)\big]\right\}.$$

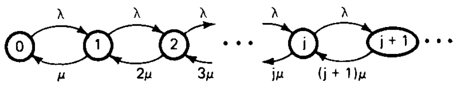

Fig. 1. Markov chain governing network state.

In the next section, we derive the distributions of N^* and N_* for variable-length packets with exponentially distributed lengths.

III. Distributions of N^* and N_* for Variable-Length Packets

Since the arrival process for packets into the network is Poisson, the interarrival times are exponentially distributed. Departures are defined to occur on completion of packets, so that the service time for a packet (equal to the packet duration) is also exponentially distributed. A packet is transmitted as soon as it is generated. Thus the entire network can be viewed as an $M/M/\infty$ queue [9], with arrivals occurring at a rate λ and departures occurring at a rate $p\mu$, where p is the number of packets currently being transmitted. The network state is defined as the number of active terminals, which is equal to the number of packets being transmitted. This is a continuous-time Markov chain; its state diagram is shown in Fig. 1. The system is assumed to be in steady state, and retransmissions are included in the packet generation process.

It is easy to see that the steady-state distribution for the network state is Poisson with mean ρ [10]. Assume that the network is in steady state at $t=0^-$, just before the initiation of the tagged packet. Since the tagged packet is initiated at $t=0$, the system state at $t=0^-$ determines the number of interfering transmissions present at the initiation of the tagged packet. Thus the random variable $N(0^+)$ is also Poisson with mean ρ, since the network is assumed to be in steady state at $t=0$.

For any event E, let $P(E|k)$ denote the conditional probability E given that $N(0^+)=k$. We find the probabilities $P(A_j|k)$ and $P(B_j|k)$ for $A_j=\{N^*\geq j\}$ and $B_j=\{N_*\leq j\}$. The computation of these conditional probabilities involves tracing the evolution of the tagged packet using an auxiliary Markov chain. This approach is based on the technique in [3]. In addition to the states corresponding to the number of interfering transmissions, there are two absorbing states [4]. One is the END state, which corresponds to the completion of the tagged packet. The other is the state A_j or B_j, respectively, depending on which of the probabilities $P(A_j)$ or $P(B_j)$ we are trying to compute. These auxiliary Markov chains are shown in Figs. 2 and 3.

For both auxiliary Markov chains, there are two possibilities for a transition from one nonabsorbing state to another. One is a packet arrival, and the other is the departure of some packet other than the tagged packet. Given that the (nonabsorbing) state k is the initial state, the (conditional) probabilities for these transitions, de-

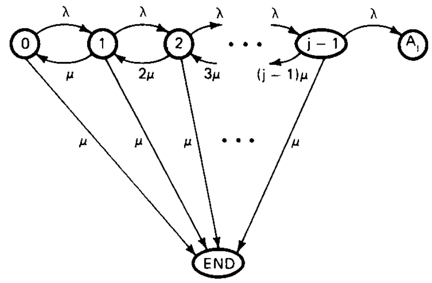

Fig. 2. Auxiliary Markov chain for N^*.

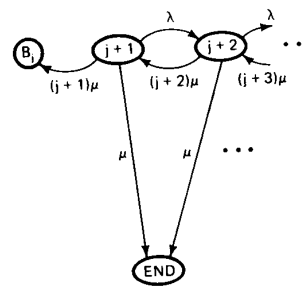

Fig. 3. Auxiliary Markov chain for N_*.

noted, respectively, by $P_k(A)$ and $P_k(D)$, are given by

$$P_k(A)=\frac{\lambda}{(k+1)\mu+\lambda}=\frac{\rho}{k+1+\rho} \tag{3.1a}$$

$$P_k(D)=\frac{k\mu}{(k+1)\mu+\lambda}=\frac{k}{k+1+\rho}. \tag{3.1b}$$

It is worth noting that the foregoing probabilities do not sum to one because there is a nonzero probability of a transition to the absorbing state END, which corresponds to the departure of the tagged packet.

The memoryless property of the exponential distribution can be exploited to write recursive relations for $P(A_j|k)$ and $P(B_j|k)$, where j is fixed and the recursion is on k. Thus, if there is a transition from state k to state $(k+1)$, the situation after the transition is exactly the same as if the initial state were $N(0^+)=k+1$. A similar result holds if there is a transition from state k to state $(k-1)$. This leads to the following recursive relations. The relations for $P(A_j|k)$ assume that $j\geq 1$, but the final expression
for $P(A_j)$ is consistent with the trivial observation that $P(A_0)=P(N^*\geq 0)=1$:

$$P(A_j|0)=P_0(A)P(A_j|1), \qquad k=0, \tag{3.2a}$$

$$P(A_j|k)=P_k(A)P(A_j|k+1)+P_k(D)P(A_j|k-1), \qquad 1\leq k\leq j-1, \tag{3.2b}$$

$$P(A_j|k)=1, \qquad k\geq j, \tag{3.2c}$$

and

$$P(B_j|k)=1, \qquad k\leq j, \tag{3.3a}$$

$$P(B_j|k)=P_k(A)P(B_j|k+1)+P_k(D)P(B_j|k-1), \qquad k\geq j+1. \tag{3.3b}$$

Note that $P(A_j|k) = P(N^* \geq j|k) = 1$ for $k \geq j$, because $N(0^+) = k$ implies that $N^* \geq N(0^+) \geq j$, and $P(B_j|k) = P(N_* \leq j|k) = 1$ for $k \leq j$, because $N(0^+) = k$ implies that $N_* \leq N(0^+) \leq j$.

The auxiliary Markov chain for N^* was previously introduced in [16], and a formulation equivalent to (3.2) is given there, in a somewhat different context. However, no explicit analytical formula for $P(A_j)$ in terms of j, ρ, f_ρ, and F_ρ is given in [16]. For our asymptotic results we do need such a formula, as well as a corresponding expression for $P(B_j)$. Thus it is necessary to solve (3.2) and (3.3) explicitly, and subsequently remove the conditioning on k. To this end we introduce some notational simplification. In particular, because j is fixed, the dependence on j is suppressed.

Define $x_k = P(A_j|k)$ and $y_k = P(B_j|k)$. Both (x_k) and (y_k) satisfy second-order difference equations. Using (3.1) and (3.2), we obtain

$$(k+1+\rho)x_k = \rho x_{k+1} + kx_{k-1}, \qquad 0 \leq k \leq j-1. \quad (3.4)$$

The equation for $k=0$ has been put in the same form as the others by assuming $x_{-1} = 0$. Thus (3.4) can be solved for $\{x_k, 0 \leq k \leq j-1\}$, using the conditions $x_{-1} = 0$, $x_j = 1$. We also have $x_k = 1$ for all $k > j$.

For $\{y_k, k \geq j+1\}$ we obtain from (3.1) and (3.3) that

$$(k+1+\rho)y_k = \rho y_{k+1} + ky_{k-1}, \qquad k \geq j+1. \quad (3.5)$$

One boundary condition is $y_j = 1$. The other is $\lim_{k\to\infty} y_k = 0$. This is proven in Appendix I by considering the interpretation of y_k in terms of N_*. Further, $y_k = 1$ for all $k < j$.

The sequences (x_k) and (y_k) can be solved for completely given the foregoing information. The following results are derived in Appendix I:

$$f_\rho(k)x_k = \frac{F_\rho(k)}{F_\rho(j)} f_\rho(j), \qquad 0 \leq k \leq j, \quad (3.6a)$$

$$x_k = 1, \qquad k \geq j, \quad (3.6b)$$

$$f_\rho(k)y_k = \frac{1-F_\rho(k)}{1-F_\rho(j)} f_\rho(j), \qquad k \geq j, \quad (3.7a)$$

and

$$y_k = 1, \qquad k \leq j. \quad (3.7b)$$

Thus,

$$P(A_j) = \sum_{k=0}^{\infty} P(A_j|k)P(N(0^+)=k) = \sum_{k=0}^{\infty} x_k f_\rho(k).$$

Similarly,

$$P(B_j) = \sum_{k=0}^{\infty} y_k f_\rho(k).$$

Now, $F^*(j) = 1 - P(N^* \geq j+1) = 1 - P(A_{j+1})$ and $F_*(j) = P(N_* \leq j) = P(B_j)$.

Using this, we can readily obtain expressions for the distribution functions of N^* and N_*, which are

$$F^*(j) = F_\rho(j) - f_\rho(j+1)\frac{F_\rho(j)}{F_\rho(j+1)}\left(j+1-\rho+\frac{\rho f_\rho(j)}{F_\rho(j)}\right) \quad (3.8)$$

and

$$F_*(j) = F_\rho(j) + f_\rho(j)\left(\rho - j - 1 + \frac{\rho f_\rho(j)}{1-F_\rho(j)}\right). \quad (3.9)$$

The corresponding probability mass functions are

$$f^*(j) = f_\rho(j)\left(j+1-\rho+\frac{\rho f_\rho(j)}{F_\rho(j)}\right) \cdot \left(1 - \frac{\rho}{j+1} + \frac{\rho}{j+1}\frac{f_\rho(j+1)}{F_\rho(j+1)}\right),$$

and

$$f_*(j) = f_\rho(j)\left(\rho - j + \frac{\rho f_\rho(j)}{1-F_\rho(j)}\right) \cdot \left(1 - \frac{j}{\rho} + \frac{f_\rho(j-1)}{1-F_\rho(j-1)}\right).$$

The results for the distributions of N^* and N_* are used to derive the asymptotic results of the next section, where we examine the system performance for both high traffic levels and large code block lengths. These results can be expressed in a general framework encompassing both fixed- and variable-length packets and both slotted and unslotted transmission, since they have the same form for all the cases considered.

IV. General Results

We will examine the behavior of the system as the offered traffic ρ and the code block length n become large. As $\rho \to \infty$, the multiple-access capability of the system must be increased to compensate for the increase in traffic level; otherwise, the trivial result that $P_E \to 1$ is obtained. Hence the ratio ρ/q is fixed to be a positive constant t, so that $q = \rho/t$. Let $\lim_\rho$ denote the limit as $\rho \to \infty$ with $\rho/q = t$ held constant.

The expressions for P_E^S, P_E^U, and P_E^L are each of the form

$$Q = \sum_{j=0}^{\infty} P_E(j)g(j) \quad (4.1)$$

where g is the appropriate probability mass function (i.e., g is f_ρ, f^*, or f_*). Let $g(x) = 0$ if x is not an integer, and denote the corresponding distribution function by G.

Substituting for $P_E(j)$ in (4.1) yields

$$Q = \sum_{j=0}^{\infty}\sum_{i=e+1}^{n}\binom{n}{i}\left[1-(1-P_h)^j\right]^i(1-P_h)^{j(n-i)}g(j).$$

An expansion of $[1-(1-P_h)^j]^i$ gives

$$Q = \sum_{j=0}^{\infty}\sum_{i=e+1}^{n}\binom{n}{i}\sum_{l=0}^{i}\binom{i}{l}(-1)^l(1-P_h)^{j(n-i+l)}g(j).$$

Now, interchanging the order of summations, we have

$$Q=\sum_{i=e+1}^{n}\binom{n}{i}\sum_{l=0}^{i}\binom{i}{l}(-1)^{l}\sum_{j=0}^{\infty}(1-P_h)^{j(n-i+l)}g(j). \tag{4.2}$$

In the foregoing g depends on ρ alone. In (4.2) only the innermost summation depends on ρ. Thus we need to consider only the limit of this summation as $\rho\to\infty$.

Define $\beta=(1-P_h)^{n-i+l}$, where the dependence on i and l is suppressed for notational convenience. Now, $P_h=cq^{-1}+O(q^{-2})$. Since $\rho/q=t$, we have $P_h=ct\rho^{-1}+O(\rho^{-2})$. Note that $P_h\to 0$ as $\rho\to\infty$, so that $\beta\to 1$ (from (4.3)) as $\rho\to\infty$. Also, $\rho P_h=ct+O(\rho^{-1})$, so that $\rho P_h\to ct$ as $\rho\to\infty$. Hence $\rho(1-\beta)\to ct(n-i+l)$ as $\rho\to\infty$, since

$$\rho(1-\beta)=\rho\left[1-(1-P_h)^{n-i+l}\right]=\rho P_h(n-i+l)\left[1+O(P_h)\right].$$

Thus, in evaluating the limits of the innermost summation, we take $\rho\to\infty$, $\beta\to 1$, with

$$\beta=1-\frac{ct(n-i+l)}{\rho}+o(\rho),\qquad \rho\to\infty. \tag{4.3}$$

The proof of the following lemma is given in Appendix II.

Lemma 1: $\lim_{\rho\to\infty}G(a\rho)=1$ for $a>1$ and $\lim_{\rho\to\infty}G(a\rho)=0$ for $0\le a<1$.

We note for future use that this implies that $\lim_{\rho\to\infty}g(\lfloor a\rho\rfloor)=0$, where $\lfloor x\rfloor$ denotes the integer part of the real number x.

In these results, g can be f_ρ, f^*, or f_* and the packets can be of either fixed or variable length. In fact, all the results in this section are this general. So, roughly speaking, Lemma 1 tells us that, for large values of the offered traffic ρ, the interference level over the duration of a packet is approximately equal to ρ for both slotted and unslotted systems. It is seen in the following that this is the key to proving that the asymptotic performance of these systems is the same in terms of both probability of error and normalized throughput.

For a slotted network [11],

$$\lim_{\rho}\sum_{j=0}^{\infty}\beta^{j}f_\rho(j)=e^{-ct(n-i+l)}.$$

Using (4.3) and Lemma 1, we will prove the more general result

$$\lim_{\rho}\sum_{j=0}^{\infty}\beta^{j}g(j)=e^{-ct(n-i+l)}. \tag{4.4}$$

The first of our main asymptotic results can now be stated as follows.

Theorem 1: For unslotted transmission of either fixed- or variable-length packets, the upper and lower bounds on the probability of error converge, as the normalized traffic level increases, to the same value as for the slotted transmission [11]. Denote this limiting value by P_1. Then

$$P_1=\lim_{\rho}P_E^U=\lim_{\rho}P_E^L=\lim_{\rho}P_E^S=\sum_{i=e+1}^{n}\binom{n}{i}(1-e^{-ct})^{i}(e^{-ct})^{n-i}. \tag{4.5}$$

Proof: Assume, for the moment, that (4.4) is true. From (4.2), we have

$$P_1=\lim_{\rho}Q=\sum_{i=e+1}^{n}\binom{n}{i}\sum_{l=0}^{i}\binom{i}{l}(-1)^{l}e^{-ct(n-i+l)}=\sum_{i=e+1}^{n}\binom{n}{i}(e^{-ct})^{n-i}(1-e^{-ct})^{i}$$

since

$$\sum_{l=0}^{i}\binom{i}{l}(-1)^{l}(e^{-ct})^{l}=(1-e^{-ct})^{i}.$$

This gives the expression for the limit in all the cases considered. It remains to prove (4.4). Let $a<1<b$.

$$\sum_{j=0}^{\infty}\beta^{j}g(j)\ge\sum_{\lfloor a\rho\rfloor+1}^{\lfloor b\rho\rfloor}\beta^{j}g(j)\ge\beta^{b\rho}\left[G(b\rho)-G(a\rho)\right], \tag{4.6}$$

where we have used the fact that $0\le\beta\le 1$. Similarly,

$$\sum_{j=0}^{\infty}\beta^{j}g(j)\le G(a\rho)+\beta^{a\rho-1}\left[G(b\rho)-G(a\rho)\right]+(1-G(b\rho)). \tag{4.7}$$

From (4.3), $\lim_\rho\beta=1$ and $\lim_\rho\beta^\rho=e^{-ct(n-i+l)}$. From Lemma 1, $G(a\rho)\to 0$ and $G(b\rho)\to 1$. Taking limits in (4.6) and (4.7), therefore, we obtain

$$e^{-bct(n-i+l)}\le\lim_{\rho}\sum_{j=0}^{\infty}\beta^{j}g(j)\le e^{-act(n-i+l)}$$

where $a<1$ and $b>1$ are arbitrary. Letting $a\to 1$ from below, and $b\to 1$ from above, we obtain the desired result (4.4). This completes the proof of Theorem 1. □

It is seen from (4.5) that the effective symbol erasure probability for high traffic levels is $(1-e^{-ct})$, which has an interesting interpretation. Even for unslotted systems, it is a good approximation to assume that the interference level remains constant over the duration of a single symbol. However, the interference level at any given time is governed by a Poisson distribution with mean ρ. The average symbol erasure probability is thus given by

$$\epsilon=\sum_{j=0}^{\infty}\left[1-(1-P_h)^{j}\right]f_\rho(j).$$

However, $P_h=cq^{-1}+O(q^{-2})=ct\rho^{-1}+O(\rho^{-2})$, which yields $\lim_{\rho\to\infty}\epsilon=1-e^{-ct}$. Hence the average symbol erasure probability is the effective erasure probability for high traffic levels. This is consistent with our earlier interpretation that the interference level is approximately constant at ρ over the duration of a packet.

TABLE I
PERFORMANCE OF SLOTTED AND UNSLOTTED SYSTEMS FOR FINITE OFFERED TRAFFIC[a]

	Fixed-length packets		Variable-length packets		
ρ	P_E^S	P_E^L	P_E^U	P_E^L	P_E^U
150	7.70E−3	3.05E−3	1.51E−2	4.03E−3	1.37E−2
300	6.88E−3	3.56E−3	1.16E−2	4.25E−3	1.07E−2
450	6.61E−3	3.87E−3	1.03E−2	4.43E−3	9.57E−3
600	6.47E−3	4.08E−3	9.58E−3	4.57E−3	8.97E−3

[a] $t = \rho/q = 0.3$, asymptotic value $P_1 = 6.08E-3$.

Since the asymptotic results are the main focus of this paper, we will restrict ourselves to the following general observations about the numerical results for *finite* values of the offered traffic ρ. For a (32,12) RS code, the convergence of the bounds on P_E to the asymptotic value P_1 (see Theorem 1) is rather slow, for both fixed- and variable-length packets. For a given ρ, the percentage error from P_1 decreases with the parameter t. For t in the 0.1–0.5 range, the bounds are within an order of magnitude of P_1 for $\rho > 150$, so that P_1 is a good measure of the performance of the system for reasonably large values of the offered traffic. The error probability P_E^S for the slotted system converges faster to P_1 and lies between the bounds for the unslotted system. In all the cases considered the lower bound P_E^L is, for a given ρ, closer to both P_1 and P_E^S than the upper bound P_E^U. Some typical results for both slotted and unslotted systems are shown in Table I. For unslotted systems with $\rho = 600$, the upper and lower bounds (for both fixed- and variable-length packets) differ by approximately a factor of two and therefore yield a fairly good estimate of the system performance. We also conjecture that the performance of the slotted system is a good approximation for that of the unslotted system for large ρ.

In the following the block codes used are (n,k) Reed–Solomon codes [2], which have the property that $e = n-k$. The code rate is $r = k/n$. We examine the system performance as $n \to \infty$ with r fixed, and we denote the corresponding limit by $\lim_n$. We will use a result from [8] which is rephrased in the following.

Fact 1: For $r, p \in [0,1]$, and n a positive integer,

$$\lim_n \sum_{i=n(1-r)+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \begin{cases} 0, & r < 1-p \\ 0.5, & r = 1-p \\ 1, & r > 1-p. \end{cases}$$

Although the proof is not provided in [8], the result can be shown easily by using standard tools such as the Chebyshev inequality and the central limit theorem.

For a Reed–Solomon code, it follows from (4.5) and Fact 1 that the limit P_{12} is

$$P_{12} = \lim_n P_1 = \begin{cases} 0, & r < e^{-ct} \\ 1, & r > e^{-ct}. \end{cases} \tag{4.8}$$

In fact, $P_{12} = 0.5$ for $r = e^{-ct}$, but this is of no consequence for the results of interest.

The same result is obtained if we interchange the order of the limits involved in evaluating P_{12}. To show this, consider the expressions for P_E^S, P_E^U, and P_E^L, and let $n \to \infty$ with $k/n = r$ kept fixed.

From (2.1), we have

$$P_E(j) = \sum_{i=n-k+1}^{n} \binom{n}{i} p_j^i (1-p_j)^{n-i}$$

with $p_j = 1-(1-P_h)^j$. Using Fact 1, we easily see that

$$\lim_n P_E(j) = \begin{cases} 0, & r < (1-P_h)^j \\ 0.5, & r = (1-P_h)^j \\ 1, & r > (1-P_h)^j. \end{cases} \tag{4.9}$$

We have $0 \le P_E(j) \le 1$ for all j, so that $0 \le P_E(j)g(j) \le g(j)$ for all j. Also, $\sum_{j=0}^{\infty} g(j) = 1 < \infty$, and g is independent of the code used. Hence the dominated convergence theorem [15] guarantees that

$$\lim_n Q = \sum_{j=0}^{\infty} g(j) \lim_n P_E(j).$$

Using (4.9), we have

$$\lim_n Q = 0.5 g\left(\frac{\ln r}{\ln(1-P_h)}\right) + 1 - G\left(\frac{\ln r}{\ln(1-P_h)}\right). \tag{4.10}$$

Thus the value of the limit need not be the same for all the cases under consideration. However, if we now let $\rho \to \infty$ with $\rho/q = t$, the result is, in all cases, identical. This is shown in the following. Substituting for q in the expression for P_h, we obtain $P_h = ct\rho^{-1} + O(\rho^{-2})$, which implies

$$\ln(1-P_h) = \ln\left(1 - ct\rho^{-1} + O(\rho^{-2})\right) = ct\rho^{-1} + O(\rho^{-2}).$$

It follows that

$$\frac{\ln r}{\ln(1-P_h)} = \frac{\ln r}{ct\rho^{-1} + O(\rho^{-2})} = \frac{a\rho}{1 + O(\rho^{-1})}$$

where

$$a = \frac{\ln(1/r)}{ct} = \frac{\ln(1/r)}{\ln e^{ct}}.$$

Thus $a > 1$ for $r < e^{-ct}$, and $a < 1$ for $r > e^{-ct}$. An application of Lemma 1 to (4.10) yields

$$\lim_\rho \lim_n Q = \begin{cases} 0, & r < e^{-ct} \\ 1, & r > e^{-ct}, \end{cases}$$

which is exactly the same result as in (4.8). Hence, using the Reed–Solomon coded FH system under consideration, $r^* = e^{-ct}$ is the highest rate at which we can communicate under high traffic levels while achieving arbitrarily small error probabilities. This result holds for both fixed- and variable-length packets.

Let us now define another performance measure, the normalized throughput. We have to be careful here, since our analysis considers codeword error probabilities rather than packet error probabilities. If a packet contains more than one codeword, it is said to be divided into frames, with one codeword per frame [11]. The throughput is defined as the expected number of successful frames per frame interval, for both fixed- and variable-length packets. This is multiplied by the rate of the code and divided by the number of frequency slots to yield the normalized throughput. As before, the network is assumed to be in steady state.

Assume, for simplicity, that each frame is of unit length. Thus the length T of a packet is equal to the number of frames it contains. This highlights a problem with the exponential distribution as a model for packet lengths, since, with probability one, the packet length is not an integer. However, we persist with this model for variable-length packets for analytical tractability. A packet is said to be successful if all the codewords it contains are decoded correctly, and the probability of success for a packet is denoted by P_S. The probability that a given codeword is decoded correctly is denoted by P_C, where $P_C = 1 - P_E$. The completion of a packet transmission is said to be a departure. Let $X(t)$ be the number of *successful* departures in the interval $[0,t]$. Let $Y(t)$ be the throughput due to these packets; that is,

$$Y(t) = \sum_{i=1}^{X(t)} T_i$$

where T_i is the length of the ith successful departure. Since the system is in steady state, $Y(t)$ has stationary increments, so that

$$E\{Y(t+s)\} = E\{Y(t)\} + E\{Y(s)\}, \qquad t \geq 0,\ s \geq 0.$$

The only measurable solution to this equation is [13] $E\{Y(t)\} = vt$, $t \geq 0$, for some $v \in [0,\infty]$. The normalized throughput can then be defined as

$$S = rv/q \tag{4.11}$$

where we have normalized with respect to both time and bandwidth.

To compute v, we use

$$v = \lim_{t \to 0} \frac{E\{Y(t)\}}{t}. \tag{4.12}$$

The arrival process in the network is Poisson with rate λ. The network can be viewed as an $M/D/\infty$ queue for fixed-length packets and as an $M/M/\infty$ queue for variable-length packets. In either case, in steady state, the departure process is also Poisson with the same rate λ. The contribution to $E\{Y(t)\}$ due to more than one departure occurring in the interval $[0,t]$ is therefore bounded by

$$\sum_{k=2}^{\infty} kE\{T\} \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \lambda t(1-e^{-\lambda t})E\{T\} = o(t), \qquad t \to 0.$$

For computing v using (4.12), therefore, it suffices to consider the event in which there is only one departure in $[0,t]$. This corresponds to computing the throughput s due to a typical packet and multiplying it by the arrival rate of packets in the network, since

$$E\{Y(t)\} = P(1 \text{ departure in } [0,t])s + o(t) = \lambda t e^{-\lambda t} s + o(t),$$

as $t \to 0$, so that, from (4.12),

$$v = \lambda s. \tag{4.13}$$

This formula holds for both fixed- and variable-length packets. As before, we are interested in asymptotic results for large traffic levels and large code block lengths. We already have asymptotic results for the codeword error probability P_E. For fixed-length packets with a single codeword in each packet, we have $s = P_C$, which yields the formula

$$S = \frac{r\lambda}{q} P_C = \frac{r\lambda}{q}(1-P_E). \tag{4.14}$$

In this special case it is easy to see that the asymptotic results for P_E give rise to corresponding results for the normalized throughput S. Another point to note in this case is that the normalized throughput can also be obtained by computing the throughput per slot. If a slot contains j packets, each packet sees $(j-1)$ interfering packets, so that, removing the conditioning on the number of packets, we have

$$S = (r/q) \sum_{j=1}^{\infty} j f_\rho(j) P_C(j-1) \tag{4.15}$$

which we may readily reduce to (4.14) by using the assumption of Poisson traffic. Specifically, we use $jf_\rho(j) = \rho f_\rho(j-1)$, and note that $P_C = \sum_{j=0}^{\infty} f_\rho(j) P_C(j)$. For unslotted systems a slotwise computation of throughput along the lines of (4.15) clearly is impossible, because the number of packets being transmitted need not remain constant over any given interval of time. From (4.11) and (4.13), we have $S = r\lambda s/q$ as our definition of normalized throughput, which generalizes the expression in (4.14). The reason that (4.14) does not apply directly is that, in general, a packet may consist of more than one codeword. Thus, as we shall see in the following, it is necessary to be more careful in deriving results for the normalized throughput from the results already obtained for the codeword error probability.

For fixed-length packets it is assumed that every packet contains m codewords, where m is a positive integer. Thus the offered traffic is

$$\rho = \lambda T = \lambda m. \tag{4.16}$$

We will use $\lim_\rho$ to denote limits as $\rho \to \infty$ with m fixed and $q = \rho/t$. For variable-length packets, the packet lengths are exponentially distributed with mean $1/\mu$, which is also equal to the average number of codewords in a packet. The offered traffic is

$$\rho = \lambda E\{T\} = \lambda/\mu \tag{4.17}$$

and $\lim_\rho$ denotes limits as $\rho \to \infty$ with $1/\mu$ fixed and $q = \rho/t$. As before, $\lim_n$ denotes limits as $n \to \infty$ with the rate fixed at r. We then have the following asymptotic result.

Theorem 2: For unslotted transmission of either fixed-length or variable-length packets, the normalized throughput S converges, for large offered traffic and large code blocklengths, as follows:

$$\lim_{\rho}\lim_{n} S = \lim_{n}\lim_{\rho} S = \begin{cases} rt, & r < e^{-ct} \\ 0, & r > e^{-ct}. \end{cases}$$

Proof: For fixed-length packets the probability of success P_S for a typical packet is bounded from above by the probability of correct decoding P_C for a typical codeword; that is, $P_S \le P_C$. A packet is unsuccessful when at least one of its codewords is decoded incorrectly, and the probability of this event can be bounded from above using a union bound as follows: $1 - P_S \le m(1 - P_C)$. This yields upper and lower bounds on P_S in terms of P_C. We have $s = mP_S$, so that, from (4.13) and (4.16), we have $v = \rho P_S$. Thus the normalized throughput is given by $S = r\rho P_S / q$. The bounds on P_S lead to corresponding bounds on S, with upper bound $S^U = r\rho P_C / q$ and lower bound $S^L = r\rho[1 - m(1 - P_C)]/q$. We know that, for large ρ and n, if $r < e^{-ct}$, then $P_C \to 1$. In this case both S^U and S^L converge to the same limit, namely rt. Hence this is also the limiting value of S. If $r > e^{-ct}$, then $P_C \to 0$. Here we have $S^U \to 0$, and $S^L \to rt(1-m) \le 0$. Since $S \ge 0$, this implies that $S \to 0$. This proves Theorem 2 for fixed-length packets.

For variable-length packets, the throughput due to a typical packet is given by

$$s = \int_0^\infty \tau P_{S|\tau} \mu e^{-\mu\tau}\, d\tau$$

where $P_{S|\tau}$ denotes the conditional probability of success for a packet of length τ. Using (4.11), (4.13), and (4.17), the following expression is obtained for the normalized throughput:

$$S = r\frac{\rho}{q}\int_0^\infty \mu\tau P_{S|\tau}\mu e^{-\mu\tau}\, d\tau.$$

We can bound $P_{S|\tau}$ using $P_{C|\tau}$, the conditional probability of correct decoding for a codeword contained in a packet of length τ. For $\tau \le 1$ we assume that the (conditional) probability of success for the packet is equal to the (conditional) probability of correct decoding for a single codeword. For $\tau \ge 1$ we proceed as we did for fixed-length packets to develop bounds on $P_{S|\tau}$. Thus we have $P_{S|\tau} = P_{C|\tau}$, $\tau \le 1$, and $1 - \tau(1 - P_{C|\tau}) \le P_{S|\tau} \le P_{C|\tau}$, $\tau \ge 1$. This yields upper and lower bounds S^U and S^L on S as follows:

$$S^U = r\frac{\rho}{q}\int_0^\infty \mu\tau P_{C|\tau}\mu e^{-\mu\tau}\, d\tau \tag{4.18}$$

and

$$S^L = r\frac{\rho}{q}\int_0^1 \mu\tau P_{C|\tau}\mu e^{-\mu\tau}\, d\tau + r\frac{\rho}{q}\int_1^\infty \mu\tau\left[1 - \tau(1 - P_{C|\tau})\right]\mu e^{-\mu\tau}\, d\tau. \tag{4.19}$$

To prove our asymptotic result, we need the following strengthened form of Lemma 1:

$$\lim_{\rho\to\infty} F^*(a\rho|\tau) = \lim_{\rho\to\infty} F_*(a\rho|\tau) = \begin{cases} 1, & a > 1 \\ 0, & 0 \le a < 1. \end{cases} \tag{4.20}$$

We prove this for the conditional distribution of N^*. The result for N_* follows in a similar fashion. The maximum interference level over the duration of a packet clearly is stochastically increasing with the packet length τ; that is, for any real x, $F^*(x|\tau)$ is nonincreasing in τ. Also,

$$F^*(a\rho|\tau = 0) = F(a\rho), \tag{4.21}$$

since the interference level at the beginning of a packet has a Poisson distribution with mean ρ. Removing the conditioning on the packet length, we have

$$F^*(a\rho) = \int_0^\infty F^*(a\rho|\tau)\mu e^{-\mu\tau}\, dt. \tag{4.22}$$

We know from Lemma 1 that $\lim_{\rho\to\infty} F^*(a\rho) = \lim_{\rho\to\infty} F(a\rho)$. The monotonicity of $F^*(x|\tau)$ in τ, together with (4.21) and (4.22) now yield the desired result (4.20).

Given (4.20), we can show, as in our earlier asymptotic results, that

$$\lim_{\rho}\lim_{n} P_{C|\tau} = \lim_{n}\lim_{\rho} P_{C|\tau} = \begin{cases} 1, & r < e^{-ct} \\ 0, & r > e^{-ct}. \end{cases}$$

This can now be used to evaluate the limiting value of the normalized throughput. The integrands in the expressions (4.18) and (4.19) for S^U and S^L are dominated by integrable functions of τ that are independent of ρ and n, since $P_{C|\tau} \le 1$, and $|1 - \tau(1 - P_{C|\tau})| \le 1 + \tau$. The dominated convergence theorem [15] can now be applied to conclude that, for $r < e^{-ct}$,

$$\lim_{\rho}\lim_{n} S^U = \lim_{n}\lim_{\rho} S^U = rt\int_0^\infty \mu\tau\mu e^{-\mu\tau}\, d\tau = rt.$$

Similarly, $\lim_\rho \lim_n S^L = \lim_n \lim_\rho S^L = rt$. For $r > e^{-ct}$, we have $\lim_\rho \lim_n S^U = \lim_n \lim_\rho S^U = 0$, and

$$\lim_{\rho}\lim_{n} S^L = \lim_{n}\lim_{\rho} S^L = rt\int_1^\infty \mu\tau[1-\tau]\mu e^{-\mu\tau}\, d\tau < 0.$$

Because $S \ge 0$, the foregoing imply the desired result for the limiting value of S. This concludes the proof of Theorem 2. □

Let the limiting value of S in Theorem 2 be denoted by $S'(r,t)$, and let $S^* = \sup_{r,t} S'(r,t)$. Then

$$S^* = \max_{t \ge 0} te^{-ct} = e^{-1}/c.$$

Note that S^* is attained using codes with rates approaching e^{-1} (since $t = 1/c$ is the maximizing argument in these equations). For slotted systems with static traffic conditions, it is shown in [8] that the asymptotic optimal code rate is a good approximation for finite block lengths as well. We believe that this holds for the kinds of systems considered here, but will not pursue this matter further in this paper.

For random hopping using memoryless frequency-hop patterns, we have $c = 2$ [6], [11]. In this case $S^* = e^{-1}/2$, which is the normalized asymptotic sum capacity of the channel [7]. Note that this is also the capacity of an unslotted Aloha channel. Moreover, if it is assumed that the dwell intervals are aligned, then $c = 1$, so that $S^* = e^{-1}$, the capacity of a slotted Aloha channel. However, if the frequency hopping is asynchronous, it does not make any difference whether slotted or unslotted transmissions are employed: the maximum throughput is $e^{-1}/2$ in either case.

We must note that it is not entirely correct to have evaluated the $\lim_\rho$ and $\lim_n$ sequentially, as we have done. Actually, as the code block length increases, so should the packet lengths, if the number of codewords per packet (or its expected value, in the case of variable-length packets) is to remain constant. Our attempt to frame the problem so as to take this into account has led to analytical intractability. For instance, for fixed-length packets, we could try to relate ρ and n more realistically as follows. Consider each packet to have length T and correspond to a single codeword of block length n. We assume a constant symbol rate of r_s symbols per second over the channel, so that $T = n/r_s$. We fix λ, the packet arrival rate in the network, and let $\rho = \lambda T$ as before. Then $\rho = \lambda n/r_s$, and we can let $n \to \infty$ and $\rho \to \infty$ in a physically meaningful way. Of course, we also let $\rho/q = t$, and keep t fixed as $\rho \to \infty$. The asymptotic analysis that must be carried out in this case seems to be difficult because of the coupling of these variables. We leave it, therefore, as an open problem.

V. Conclusion

The asymptotic performance of slotted and unslotted systems is seen to be identical for high traffic levels. Although our model is idealized, this does indicate that frequency hopping reduces the effect of fluctuations in the multiple-access interference due to a lack of slotting in the network. Our results also indicate that the performance of frequency-hop multiple-access is relatively insensitive to the distribution of packet lengths, since all our asymptotic results hold for both fixed and exponentially distributed packet lengths. We have also shown that the asymptotic sum capacity of the FH channel can be attained using Reed–Solomon coding. Two qualifications must be made regarding this result, however. Although the asymptotic efficacy of Reed–Solomon codes has been demonstrated both by our results and by earlier work [8], it must be noted that the size of the alphabet for the code symbols increases as the block length increases. This must be kept in mind in interpreting these asymptotic results as an argument in favor of using Reed–Solomon codes in practical systems. Secondly, our results for large block lengths and high traffic levels are not independent of the manner in which ρ and n approach infinity. This restricts the statements we can make about the achievability of throughputs that are arbitrarily close to the asymptotic maximum possible throughput and also does not account for the fact that ρ and n often cannot go to infinity independently, as discussed at the end of Section IV.

In conclusion, we note that the technique of analysis used here can be applied to incorporate the effects of thermal noise, fading, and imperfect side information [6], [13]. Finite population models with queuing at the terminals can also be analyzed using the idea of maximum and minimum interference levels [9].

Appendix I

We supply the details of the derivation of the distributions of N^* and N_* for variable-length packets in this appendix. The notation used here is the same as in Section III.

For a given j, it is required to solve (3.4) and (3.5) subject to $x_{-1} = 0$, $x_j = 1$, $y_j = 1$, and $\lim_{k\to\infty} y_k = 0$. The proof of the last boundary condition is as follows. Recall that $y_k = P(N_* \le j|k)$ and $N_* = \min\{N(t), t \in [0,T]\}$. According to our conditioning, $N(0^+) = k$. Let $\{N_-(t), t \in [0,T]\}$ be the pure death process representing the number of the k original packets that have not ended by the time t. Clearly, $N_-(t) \le N(t)$ for all t, so that

$$P\Big(\min_{0\le t\le T} N(t) \le j|k\Big) \le P\Big(\min_{0\le t\le T} N_-(t) \le j|k\Big).$$

However, $\min_{0\le t\le T} N_-(t) = N_-(T)$, since $N_-(t)$ is nonincreasing. Thus

$$y_k = P(N_* \le j|k) \le P(N_-(T) \le j|k). \tag{A.1}$$

Since the departure process for packets is Poisson,

$$P(N_-(t) = i|k) = \frac{(\mu t)^{k-i}}{(k-i)!}e^{-\mu t}, \qquad 0 \le i \le k,$$

so that

$$\begin{aligned} P(N_-(T) \le j|k) &= \int_0^\infty \sum_{i=0}^{j} P(N_-(t) = i|k)\mu e^{-\mu t}\,dt \\ &= \sum_{i=0}^{j}\int_0^\infty \frac{(\mu t)^{k-i}}{(k-i)!}e^{-\mu t}\mu e^{-\mu t}\,dt \\ &= \sum_{i=0}^{j}\frac{1}{2^{k-i+1}} \\ &= \frac{1}{2^{k+1}}(2^{j+1}-1) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Using (A.1), it follows that $\lim_{k\to\infty} y_k = 0$ for each j, which is the required result.

Next consider (3.4) over the range $0 \le k \le i$, for $i \le j-1$. By summing each side of these $(i+1)$ equations, it is found that the terms telescope to yield

$$(i+1)x_i + \rho x_0 = \rho x_{i+1}, \qquad 0 \le i \le j-1.$$

For further telescoping, the $(i+1)$th equation is multiplied by s_i, and the equations from $i = 0$ to $i = k$ are added. For telescoping to occur, we must have $(i+1)s_i = \rho s_{i-1}$. Choosing $s_0 = 1$ results in

$$s_i = \frac{\rho^i}{(i+1)!}, \qquad 0 \le i \le j-1. \tag{A.2}$$

Thus, on adding the equations using the foregoing choice of multipliers, we obtain

$$s_0 x_0 + \rho x_0 \sum_{i=0}^{k} s_i = s_k \rho x_{k+1}, \qquad 0 \le k \le j-1,$$

so that, using (A.2),

$$\frac{\rho^{k+1}}{(k+1)!} x_{k+1} = x_0 \sum_{i=0}^{k+1} \frac{\rho^i}{i!}, \qquad 0 \le k \le j-1.$$

Replacing $(k+1)$ by k and using the notation established in Section III, we obtain $f_\rho(k)x_k = F_\rho(k)x_0$, $1 \le k \le j$. However, $x_j = 1$, so that $x_0 = f_\rho(j)/F_\rho(j)$, which yields (3.6a).

Now consider (3.5). Adding the first $(i-j)$ equations, from $k = j+1$ to $k = i$, it is seen that here, too, the terms telescope to yield

$$(i+1)y_i + \rho y_{j+1} = \rho y_{i+1} + (j+1)y_j, \qquad i \ge j+1.$$

Substituting $y_j = 1$,

$$(i+1)y_i + \rho y_{j+1} = \rho y_{i+1} + j + 1, \qquad i \ge j$$

where the additional equation for $i = j$ is a trivial equality. As before, further telescoping is achieved by multiplying the equations by different weights and adding. Let r_i be the weight for the $(i-j+1)$th equation. For telescoping to occur, we need $(i+1)r_i = \rho r_{i-1}$ for $i \ge j+1$. Taking $r_j = 1$, we have

$$r_i = \frac{(j+1)!}{(i+1)!}\rho^{i-j}, \qquad i \ge j. \tag{A.3}$$

Thus, adding the $(k-j+1)$ weighted equations from $i = j$ to $i = k$ yields

$$(j+1)y_j r_j + \rho y_{j+1}\left(\sum_{i=0}^{k} r_i\right) = \rho y_{k+1} r_k + (j+1)\left(\sum_{i=0}^{k} r_i\right), \qquad k \ge j.$$

Substituting from (A.3), and putting $y_j = 1$,

$$\frac{\rho^{k+1}}{(k+1)!} y_{k+1} = \frac{\rho^j}{j!} + \left(y_{j+1} - \frac{j+1}{\rho}\right)\sum_{i=j}^{k} \frac{\rho^{i+1}}{(i+1)!}, \qquad k \ge j.$$

Replacing $(k+1)$ by k, and using our familiar notation,

$$f_\rho(k)y_k = f_\rho(j) + \left(y_{j+1} - \frac{j+1}{\rho}\right)\sum_{i=j+1}^{k} f_\rho(i), \qquad k \ge j+1. \tag{A.4}$$

Now let $k \to \infty$ to obtain (using the boundary condition that $y_k \to 0$),

$$0 = f_\rho(j) + \left(y_{j+1} - \frac{j+1}{\rho}\right)\left[1 - F_\rho(j)\right],$$

so that $\rho^{-1}(j+1) - y_{j+1} = f_\rho(j)/[1 - F_\rho(j)]$. Substituting this in (A.4) produces

$$f_\rho(k)y_k = f_\rho(j) - \frac{f_\rho(j)}{1 - F_\rho(j)}\left[F_\rho(k) - F_\rho(j)\right], \qquad k \ge j+1,$$

which simplifies to (3.7a).

This completes the solution to the second difference equation. From these results the distributions of N^* and N_* can be derived by removing the conditioning, as outlined in Section III.

Appendix II

We now prove Lemma 1, which is the key result used in deriving the asymptotic results of Section IV. The earlier notation is preserved, except that we drop the subscripts on f_ρ and F_ρ.

Consider $a \ge 0$, $a \ne 1$, and let X be a Poisson random variable with mean ρ. Then it is a standard result that $E\{X\} = \rho$ and $\sigma^2 = \operatorname{var}(X) = E\{(X - E\{X\})^2\} = \rho$. From the Chebyshev inequality,

$$P\left[|X - \rho| \ge K\sqrt{\rho}\right] \le 1/K^2.$$

For $a > 1$,

$$1 - F(a\rho) = P[X - \rho > (a-1)\rho] \le P[|X - \rho| > (a-1)\rho] \le \frac{1}{(a-1)^2\rho} \to 0 \quad \text{as} \quad \rho \to \infty.$$

Thus $\lim_{\rho\to\infty} F(a\rho) = 1$ for $a > 1$. Proceeding in similar fashion for $0 \le a < 1$, we obtain

$$\lim_{\rho\to\infty} F(a\rho) = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a > 1. \end{cases} \tag{B.1}$$

This proves Lemma 1 for F. In what follows it is proven for F^* and F_*, both for fixed- and variable-length packets. We will need the following facts.

Fact B.1: For any $a \ge 0$, define $h(a) = a\ln(a) + 1 - a$. Then $h(a) > 0$ if and only if $a \ne 1$, and

$$f(\lfloor a\rho \rfloor) \le \exp\{-\rho h(a)[1 + o(1)]\}, \qquad \rho \to \infty,$$

that is, if $a \ne 1$, then $f(\lfloor a\rho \rfloor) \to 0$ exponentially as $\rho \to \infty$.

Fact B.2: $\lim_{\rho\to\infty} \sum_{j=0}^{\infty} f^2(j) = 0$.

Fact B.3: $\lim_{\rho\to\infty} \sum_{j=0}^{\infty} f(j)F(j) = \lim_{\rho\to\infty} \sum_{j=0}^{\infty} f(j)[1 - F(j)] = 1/2$.

Fact B.1 is proven first. We note that h is convex and nonnegative over its domain of definition and has a unique global minimum at $a = 1$. Since $h(1) = 0$, we have $h(a) > 0$ for $a \ge 0$, $a \ne 1$. Now,

$$f(\lfloor a\rho \rfloor) = \frac{\rho^{\lfloor a\rho \rfloor}}{(\lfloor a\rho \rfloor)!}e^{-\rho}, \qquad a \ge 0, a \ne 1.$$

By Stirling's formula, $n! \ge n^n e^{-n}$, $n = 0, 1, 2, \cdots$, so that

$$f(\lfloor a\rho \rfloor) \le \frac{\rho^{\lfloor a\rho \rfloor}e^{-\rho}}{(\lfloor a\rho \rfloor)^{\lfloor a\rho \rfloor}e^{-\lfloor a\rho \rfloor}} = \left(\frac{\rho}{\lfloor a\rho \rfloor}\right)^{\lfloor a\rho \rfloor} e^{\lfloor a\rho \rfloor - \rho}.$$

Taking logarithms and dividing by ρ, we obtain

$$\frac{1}{\rho}\ln f(\lfloor a\rho \rfloor) \le \frac{\lfloor a\rho \rfloor}{\rho}\ln\left(\frac{\rho}{\lfloor a\rho \rfloor}\right) + \frac{\lfloor a\rho \rfloor - \rho}{\rho}.$$

Because the limit (as $\rho \to \infty$) of the upper bound is $a\ln(a^{-1}) + a - 1 = -h(a)$, we see that $\rho^{-1}\ln f(\lfloor a\rho \rfloor) \le -h(a) + o(1)$ for large ρ, which yields the desired result.

To prove Fact B.2, we express the quantities of interest in terms of the zeroth-order modified Bessel function of the first kind [1] and use the asymptotic properties of the latter:

$$\sum_{j=0}^{\infty} f^2(j) = e^{-2\rho}\sum_{j=0}^{\infty}\frac{\rho^{2j}}{j!j!} = e^{-2\rho}I_0(2\rho) \to 0 \quad \text{as} \quad \rho \to \infty.$$

For the proof of Fact B.3, note that

$$\sum_{j=0}^{\infty} f(j)F(j) + \sum_{j=0}^{\infty} f(j)[1 - F(j)] = \sum_{j=0}^{\infty} f(j) = 1. \tag{B.2}$$

Also,

$$\sum_{j=0}^{\infty} f(j)F(j) = \sum_{j=0}^{\infty} f(j)\sum_{k=0}^{j} f(k) = \sum_{k=0}^{\infty} f(k)\sum_{j=k}^{\infty} f(j)$$
$$= \sum_{k=0}^{\infty} f(k)[1-F(k)] + \sum_{k=0}^{\infty} f^2(k).$$

Taking limits as $\rho \to \infty$ on both sides, we get the desired result by using Fact B.2 and (B.2).

We now prove Lemma 1 for fixed-length packets. From [11],

$$P(N^* \ge j|k,l) = \begin{cases} 1, & 0 \le j \le \max(k,l) \\ \binom{k+l}{j}\binom{k+l}{k}^{-1}, & \max(k,l) \le j \le k+l \\ 0, & j > k+l \end{cases}$$

where k,l are independent and identically Poisson distributed (i.i.d. Poisson) with mean ρ. Removing the conditioning from this expression,

$$\begin{aligned} P(N^* \ge j) &= 2\sum_{k=j+1}^{\infty} f(k)\sum_{l=0}^{j} f(l) - \sum_{k=j+1}^{\infty} f^2(k) \\ &\quad + \sum_{k=0}^{j}\sum_{l=j-k}^{j} f(k)f(l)\binom{k+l}{j}\binom{k+l}{k}^{-1} \\ &= 2\sum_{k=j+1}^{\infty} f(k)F(k) - \sum_{k=j+1}^{\infty} f^2(k) \\ &\quad + \sum_{k=0}^{j}\sum_{l=j-k}^{j} f(j)f(k+l-j) \\ &= 2\sum_{k=j+1}^{\infty} f(k)F(k) - \sum_{k=j+1}^{\infty} f^2(k) \\ &\quad + f(j)\sum_{k=0}^{j} F(k). \end{aligned} \tag{B.3}$$

Consider $j = \lfloor a\rho \rfloor$, $a \ne 1$, $a > 0$, and note that

$$\sum_{k=j+1}^{\infty} f^2(k) < \sum_{k=0}^{\infty} f^2(k) \to 0 \quad \text{as} \quad \rho \to \infty$$

by Fact B.2, and that using Fact B.1,

$$f(\lfloor a\rho \rfloor)\sum_{k=0}^{\lfloor a\rho \rfloor} F(k) \le (\lfloor a\rho \rfloor + 1) f(\lfloor a\rho \rfloor)$$
$$\le (\lfloor a\rho \rfloor + 1)e^{-\rho h(a)(1+o(1))} \to 0 \quad \text{as} \quad \rho \to \infty.$$

This takes care of the second and third terms in (B.3). For the first term, we use the bound

$$2\sum_{k=\lfloor a\rho \rfloor+1}^{\infty} f(k)F(k) \le 2\sum_{k=\lfloor a\rho \rfloor+1}^{\infty} f(k) = 2[1 - F(\lfloor a\rho \rfloor)],$$

and use (B.1) to conclude that, for $a > 1$,

$$1 - F(\lfloor a\rho \rfloor) \to 0 \quad \text{as} \quad \rho \to \infty.$$

For $0 < a < 1$,

$$2\sum_{k=\lfloor a\rho \rfloor+1}^{\infty} f(k)F(k) = 2\sum_{k=0}^{\infty} f(k)F(k) - 2\sum_{k=0}^{\lfloor a\rho \rfloor} f(k)F(k).$$

Fact B.3 implies that

$$2\sum_{k=0}^{\infty} f(k)F(k) \to 1 \quad \text{as} \quad \rho \to \infty,$$

and using (B.1), we have

$$2\sum_{k=0}^{\lfloor a\rho \rfloor} f(k)F(k) \le 2F(\lfloor a\rho \rfloor) \to 0 \quad \text{as} \quad \rho \to \infty.$$

For $a = 0$, we have $P(N^* \ge 0) = 1$ trivially. The foregoing results can be stated together as follows:

$$P(N^* \ge \lfloor a\rho \rfloor] \to \begin{cases} 1, & 0 \le a < 1 \\ 0, & a > 1 \end{cases}$$

from which it follows that

$$F^*(a\rho) \to \begin{cases} 0, & 0 \le a < 1 \\ 1, & a > 1. \end{cases}$$

Consider the distribution of N_*. From [11],

$$P(N_* \le j|k,l) = \begin{cases} \binom{k+l}{j}\binom{k+l}{k}^{-1}, & 0 \le j \le \min(k,l) \\ 1, & j \ge \min(k,l) \end{cases}$$

where k,l are i.i.d. Poisson with mean ρ, as before. Removing the conditioning,

$$P(N_* \le j) = \sum_{l=j+1}^{\infty}\sum_{k=j+1}^{\infty} f(k)f(l)\binom{k+l}{j}\binom{k+l}{k}^{-1} + 2\sum_{l=0}^{j}\sum_{k=l+1}^{\infty} f(k)f(l) + \sum_{l=0}^{j} f^2(l). \tag{B.4}$$

Rewriting the first term, we have

$$\begin{aligned} &\sum_{l=j+1}^{\infty}\sum_{k=j+1}^{\infty} f(k)f(l)\binom{k+l}{j}\binom{k+l}{k}^{-1} \\ &= \sum_{l=j+1}^{\infty}\sum_{k=j+1}^{\infty} f(j)f(k+l-j) \\ &= f(j)\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} f(k+l+j) = f(j)\sum_{n=2}^{\infty}(n-1)f(n+j) \\ &= f(j)\sum_{n=j+2}^{\infty}(n-j-1)f(n) \\ &= f(j)\sum_{n=j+2}^{\infty} nf(n) - (j+1)f(j)\sum_{n=j+2}^{\infty} f(n). \end{aligned}$$

Put $j = \lfloor a\rho \rfloor$, $a \ne 1$, $a \ge 0$, in the foregoing. By Fact B.1,

$$f(\lfloor a\rho \rfloor)\sum_{n=\lfloor a\rho \rfloor+2}^{\infty} nf(n) \le e^{-\rho h(a)(1+o(1))}\rho \to 0 \quad \text{as} \quad \rho \to \infty,$$

and

$$(\lfloor a\rho \rfloor + 1)f(\lfloor a\rho \rfloor)\sum_{n=\lfloor a\rho \rfloor+2}^{\infty} f(n)$$
$$\le (\lfloor a\rho \rfloor + 1)e^{-\rho h(a)(1+o(1))} \to 0 \quad \text{as} \quad \rho \to \infty.$$

Thus the first term in (B.4) tends to zero as $\rho \to \infty$. We also have, by Fact B.2, that

$$\sum_{l=0}^{\lfloor a\rho \rfloor} f^2(l) \le \sum_{l=0}^{\infty} f^2(l) \to 0 \quad \text{as} \quad \rho \to \infty,$$

which takes care of the third term in (B.4). The second term in

(B.4) can be rewritten as $2\sum_{l=0}^{\lfloor a\rho \rfloor} f(l)[1-F(l)]$. For $a>1$,

$$2\sum_{l=0}^{\lfloor a\rho \rfloor} f(l)[1-F(l)] = 2\sum_{l=0}^{\infty} f(l)[1-F(l)] - 2\sum_{l=\lfloor a\rho \rfloor+1}^{\infty} f(l)[1-F(l)].$$

By Fact B.3,

$$2\sum_{l=0}^{\infty} f(l)[1-F(l)] \to 1 \quad \text{as} \quad \rho\to\infty,$$

and using (B.1),

$$2\sum_{l=\lfloor a\rho \rfloor+1}^{\infty} f(l)[1-F(l)] \le 2\sum_{l=\lfloor a\rho \rfloor+1}^{\infty} f(l) = 2[1-F(\lfloor a\rho \rfloor)] \to 0 \quad \text{as} \quad \rho\to\infty,$$

so that $2\sum_{l=0}^{\lfloor a\rho \rfloor} f(l)[1-F(l)] \to 1$ as $\rho\to\infty$. For $a<1$,

$$2\sum_{l=0}^{\lfloor a\rho \rfloor} f(l)[1-F(l)] \le 2\sum_{l=0}^{\lfloor a\rho \rfloor} f(l) = 2F(\lfloor a\rho \rfloor) \to 0 \quad \text{as} \quad \rho\to\infty,$$

using (B.1). Combining these results, we have

$$F_*(a\rho) = P(N_* \le \lfloor a\rho \rfloor) \to \begin{cases} 0, & 0\le a<1 \\ 1, & a>1. \end{cases}$$

This completes the proof of Lemma 1 for fixed-length packets.

For variable-length packets, we have from (3.8) that

$$F^*(j) = F(j) - f(j+1)\frac{F(j)}{F(j+1)}\left(j+1-\rho+\frac{\rho f(j)}{F(j)}\right).$$

The magnitude of the second term is bounded according to

$$f(j+1)\frac{F(j)}{F(j+1)}\left|j+1-\rho+\frac{\rho f(j)}{F(j)}\right| \le \frac{\rho}{j+1} f(j)|j+1+2\rho|.$$

For $j=\lfloor a\rho \rfloor$, $a\ge 0$, and $a\ne 1$, we have that the foregoing is overbounded by $e^{-\rho h(a)(1+o(1))}\rho(\lfloor a\rho \rfloor+1)^{-1}(\lfloor a\rho \rfloor+1+2\rho)$, which goes to zero as $\rho\to\infty$ using Fact B.1. Hence, using (B.1),

$$\lim_{\rho\to\infty} F^*(a\rho) = \lim_{\rho\to\infty} F(a\rho) = \begin{cases} 0, & 0\le a<1 \\ 1, & a>1. \end{cases}$$

From (3.9),

$$F_*(j) = F(j) + f(j)\left(\rho - j - 1 + \frac{\rho f(j)}{1-F(j)}\right).$$

In an entirely similar fashion, it can be shown that the second term here tends to zero as $\rho\to\infty$, so that

$$\lim_{\rho\to\infty} F_*(a\rho) = \lim_{\rho\to\infty} F(a\rho) = \begin{cases} 0, & 0\le a<1 \\ 1, & a>1. \end{cases}$$

This completes the proof of Lemma 1 for variable-length packets.

References

[1] H. Bateman, *Higher Transcendental Functions*, vol. 2. New York: McGraw-Hill, 1953.

[2] E. R. Berlekamp, "The technology of error-correcting codes," in *Proc. IEEE*, vol. 68, May 1980, pp. 564–593.

[3] J. M. Brazio and F. A. Tobagi, "Theoretical results in throughput analysis of multihop packet radio networks," in *Conf. Rec. 1980 IEEE Int. Conf. Communications*, June 1980, pp. 448–455.

[4] W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1. New York: Wiley, 1957.

[5] C. D. Frank and M. B. Pursley, "On the statistical dependence of hits in frequency-hop multiple-access," *IEEE Trans. Commun.*, to appear.

[6] E. A. Geraniotis and M. B. Pursley, "Error probabilities for slow frequency-hopped spread-spectrum multiple access communications over fading channels," *IEEE Trans. Commun.*, vol. COM-30, no. 5, pp. 996–1009, May 1982.

[7] M. Hegde and W. E. Stark, "Multiple-access capability of frequency-hop spread-spectrum communications," in *Conf. Rec., IEEE Military Communications Conf.*, Oct. 1985, pp. 575–579.

[8] S. W. Kim and W. Stark, "Optimum rate Reed-Solomon codes for frequency-hopped spread-spectrum multiple-access communication systems," *IEEE Trans. Commun.*, vol. COM-37, no. 2, pp. 138–144, Feb. 1989.

[9] B. G. Kim and W. E. Stark, "Multiple-access capability of Reed-Solomon coded frequency-hop spread spectrum communication systems," in *Proc. Allerton Conf. Communication, Control, and Computing*, 1986, pp. 530–539.

[10] L. Kleinrock, *Queuing Systems*, vol. 1. New York: Wiley, 1975.

[11] U. Madhow, "Bounds and approximations for the performance of asynchronous frequency-hop packet radio networks," Coordinated Sci. Lab., Univ. of Illinois, Urbana–Champaign, Rep. UILU-ENG-87-2236, June 1987.

[12] M. B. Pursley, "Frequency-hop transmission for satellite packet switching and terrestrial packet radio networks," *IEEE Trans. Inform. Theory*, vol. IT-32, no. 5, pp. 652–667, Sept. 1986.

[13] ——, "Tradeoffs between side information and code rate in slow frequency-hop packet radio networks," in *Conf. Rec. 1987 IEEE Int. Conf. Communications*, Seattle, WA, vol. 2, June 1987, pp. 947–952.

[14] S. M. Ross, *Applied Probability Models with Optimization Applications*. San Francisco: Holden-Day, 1970.

[15] H. L. Royden, *Real Analysis*. New York: Macmillan, 1968.

[16] F. A. Tobagi and J. S. Storey, "Improvements in throughput of a CDMA packet radio network due to a channel load sense access protocol," in *Proc. Allerton Conf. Communication, Control, and Computing*, 1984, pp. 40–49.