CAPACITY MAXIMIZATION FOR DISTRIBUTED BROADBAND BEAMFORMING

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ABSTRACT

Most prior research in distributed beamforming involves narrowband, frequency nonselective, channels, with the goal of sending a common message from cooperating nodes so that phases of the signals transmitted from the different nodes align at the receiver. The performance metric is the received SNR (directly related to the Shannon capacity for an AWGN channel). This "coherence metric" is maximized when each transmitter compensates its channel phase to the receiver, while transmitting at maximum allowable power. In this paper, we consider the problem of distributed transmit beamforming over broadband, frequency selective channels, defining the coherence metric as the Shannon capacity, to be maximized subject to a power constraint at each transmitter. OFDM provides a natural decomposition of such channels into narrowband subchannels, hence the problem reduces to determining how each transmitter allocates its power across subchannels. A key technical result is that the optimal solution obeys a separation property that significantly simplifies computation. We show that it differs from classical waterfilling due to the per-transmitter power constraints of the distributed beamforming setting. We compare it both structurally and numerically, to a centralized beamforming system with power constraint across transmitters. This is like waterfilling and upper bounds the performance of our setup.

Index Terms— Broadband, capacity, maximization, water filling, coherence.

1. INTRODUCTION

In distributed transmit beamforming over a frequency nonselective channel, cooperating transmitters convey a common message by forming a virtual array directing a beam at a receiver. This concept plays a fundamental role in the Shannon theory of relaying [1]. For an AWGN channel, maximizing Shannon capacity and received SNR are equivalent, and are achieved when each transmitter precodes so as to compensate for its channel phase to the receiver, while transmitting at its maximum power. Realizing this optimum solution in practice requires synchronizing transmitters with independent oscillators, and ensuring that their phases align at the receiver. In recent years, there has been substantial progress in solving these

difficult synchronization problems, with theoretical [4, 5] as well as experimental results [6–8] showing that feedback-based strategies yield near-optimal performance.

To the best of our knowledge, prior work on distributed transmit beamforming has been restricted to frequency nonselective channels, with the exception of [9], which presents preliminary results on extending feedback-based algorithms to broadband, frequency selective channels. In this paper, our goal is to establish fundamental performance benchmarks for wideband channels which can serve as a point of comparison for any proposed distributed transmit beamforming algorithm. In particular, we pick the simplest possible metric, Shannon capacity, and seek to optimize it subject to *per-transmitter* power constraints. Decomposing the broadband channel into N frequency nonselective subchannels via OFDM, the problem reduces to determining how each of n users allocates its power across subchannels so as to maximize capacity.

The problem we consider has an interesting form. The utility function is a sum over subchannels of logarithms whose arguments involve (per-subchannel) sums over transmitters, but the constraints are per transmitter, with each constraint involving a sum over subchannels. In this sense the nature of terms appearing in the constraint and the objective are of different types. This is in sharp relief to the vast literature on Shannon theory for MIMO systems, with strategies often reducing to some form of waterfilling. For example, the vector multiple access channel [10, 11], does have per-transmitter power constraints, but unlike here, its objective functions *also involve per-transmitter as opposed to per-subchannel terms*. As an aside, we also note that in [10, 11] transmitters send independent messages. In our setting they must send a common message.

We compare our solution to a centralized system which provides an upper bound on capacity by optimizing it subject to a looser constraint on the sum of the powers across transmitters. This centralized problem can indeed be reduced to classical water filling, since we may view each subchannel as a MISO channel with a centralized transmit array. Furthermore, just like [10, 11], and unlike distributed beamforming, this centralized structure also has the property that the same type of terms appear in both the constraints and the objective function. While the objective is the same as decentralized objective, the constraint can be treated as also being a per-transmitter summation.

Our analysis reveals that the optimal solution obeys an interesting separation property, which is aesthetically appealing, while simplifying computation: it reduces an nN-variable constrained optimization with complex variables, to an N-*variable unconstrained optimization with real variables*. A consequence of this separation property is that, if a given user is silent on a subchannel, then *all users* must be silent on that subchannel. In particular, the solution that we obtain is very different from classical waterfilling. Yet the separation property does enable our solution to inherit two properties of the centralized solution, though through very different technical mechanisms: (a) Both lead to entire subchannels being potentially silent at the optimal. (b) Both reduce to an N-variable real optimiza-

tion. However, whereas in our case *this N-variable optimization is an unconstrained problem,* that in the centralized problem remains *constrained.*

For the settings considered in our numerical results, the capacity for our distributed system with per-transmitter constraints is quite close to the upper bound provided by the centralized problem. On the other hand, capacity obtained by spreading each transmitters power evenly across subchannels is found to perform quite poorly relative to our optimal distributed system, especially at low SNRs.

Outline: Section 2 provides an OFDM-based problem formulation, and explains how our problem differs from classical water-filling. It provides without detailed derivation, the optimum for the centralized benchmark, and foreshadows its key difference with our decentralized criterion. Section 3 provides the main results, exposing the separation property. Section 4 gives simulations. Section 5 concludes.

2. THE OPTIMIZATION PROBLEMS

In Section 2.1 we formulate an optimization problem that helps quantify the best achievable distributed beamforming in an OFDM setting for both decentalized beamforming and a centralized benchmark. Section 2.2 explains the technical difference between the OFDM based criterion for distributed beamforming and those appearing in classical water filling and provides the condition of optimality for the centralized benchmark.

2.1. Criteria for broadband beamforming

Consider transmitters indexed over $i \in \{1, \cdots, n\}$, each transmitting over N subchannels indexed by $k \in \{0, \cdots, N-1\}$. Suppose the complex baseband channel seen by the i-th transmitter on the k-th subchannel is $\hat{h}_{ik}$. Suppose the noise variance on this subchannel is σ_k^2. Suppose each transmitter transmits the common complex signal x_k over the k-th subchannel, after precoding by the complex gain $\hat{g}_{ik}$. Then the aggregate received signal in the k'th subchannel is $r_k = x_k \sum_{i=1}^{n} \hat{g}_{ik}\hat{h}_{ik}$. The received signal power is then $P_{R,k} = E\left(|x_k|^2\right) \left|\sum_{i=1}^{n} \hat{g}_{ik}\hat{h}_{ik}\right|^2$. The capacity of the MISO channel can then be written as:

$$\sum_{k=0}^{N-1} \ln\left(1 + \frac{P_{x,k}\left|\sum_{i=1}^{n} \hat{g}_{ik}\hat{h}_{ik}\right|^2}{\sigma_k^2}\right) \qquad (2.1)$$

where $P_{x,k} \doteq E\left(|x_k|^2\right)$. In *distributed beamforming* the optimization goal is to find $\hat{g}_{ik}$ to maximize (2.1) subject to the total transmitted power by each transmitter being equal to unity, i.e. subject to

$$\sum_{k=0}^{N-1} P_{x,k}|\hat{g}_{ik}|^2 = 1, \ \forall\, i \in \{1, \cdots, n\} \qquad (2.2)$$

On the other hand in *centralized beamforming* transmitters can coordinate to distribute the power among themselves. Consequently optimum beamforming is to maximize (2.1) subject to

$$\sum_{i=1}^{n} \sum_{k=0}^{N-1} P_{x,k}|\hat{g}_{ik}|^2 = n. \qquad (2.3)$$

Now redefine $g_{ik} = \hat{g}_{ik}\sqrt{P_{x,k}}$ and $h_{ik} = \hat{h}_{ik}/\sigma_k$. Note $|h_{ik}|^2$ models the SNR seen by the i-th transmitter in the k-th subchannel.

Then the problem for decentralized optimal beamforming reduces to Problem 2.1 below.

Problem 2.1 *Given complex scalars h_{ik}, find complex scalars g_{ik} that maximize*

$$\sum_{k=0}^{N-1} \ln\left(1 + \left|\sum_{i=1}^{n} g_{ik}h_{ik}\right|^2\right) \qquad (2.4)$$

subject to:

$$\sum_{k=0}^{N-1} |g_{ik}|^2 = 1, \ \forall\, i \in \{1, \cdots, n\} \qquad (2.5)$$

The centralized criterion becomes with the same definitions:

Problem 2.2 *Given complex scalars h_{ik}, find complex scalars g_{ik} to maximize (2.4) subject to:*

$$\sum_{i=1}^{n} \sum_{k=0}^{N-1} |g_{ik}|^2 = n. \qquad (2.6)$$

Observe the optimum capacity for Problem 2.2 is no smaller than that for Problem 2.1.

2.2. Technical differences and the solution to Problem 2.2

The key technical difference between Problem 2.1 and papers like [10] and [11] is as follows. Summations in each logarithm term in (2.4) is with respect to n, the transmitter index, while the constraints (2.5) are summations with respect to k, the subchannel index. Comparable utility functions considered in [11] involve summations over the same index as the summations in the constraints. It is this difference that fundamentally alters the nature of the solution to Problem 1, than those offered by traditional water filling papers. Problem 2 on the other hand does not have this technical difference as the summation in (2.6) is over both indices.

For the sake of completeness we now provide the solution of Problem 2.2, omitting details due to its similarity to the traditional water filling methods. Define the vectors of precoders and SNRs over the k-th subchannel to be respectively

$$\boldsymbol{g}_k = \left[g_{1k}, \cdots, g_{nk}\right]^T \text{ and } \boldsymbol{h}_k = \left[h_{1k}, \cdots, h_{nk}\right]^T. \qquad (2.7)$$

Then (2.4) and (2.6) respectively reduce to

$$\sum_{k=0}^{N-1} \ln\left(1 + \left|\boldsymbol{h}_k^T \boldsymbol{g}_k\right|^2\right) \text{ and } \sum_{k=0}^{N-1} \|\boldsymbol{g}_k\|^2 = n.$$

Thus by the Cauchy-Schwarz inequality, for some real scalar c_k, the optimizing g_k obey:

$$g_k = c_k^2 h_k^*, \qquad (2.8)$$

and Problem 2.2 reduces to: find real scalar c_k that maximize

$$\sum_{k=0}^{N-1} \ln\left(1 + c_k^2 \|\boldsymbol{h}_k\|^2\right)$$

subject to

$$\sum_{k=0}^{N-1} c_k^4 \|\boldsymbol{h}_k\|^2 = n.$$

Effectively, this Nn variable complex optimization has been reduced to an N variable real though constrained optimization. Arrange the

h_k to obey $\|h_k\| \geq \|h_{k+1}\|$. Suppose M is the largest integer for which

$$\frac{1}{M}\left(n + \sum_{l=0}^{M-1}\frac{1}{\|h_l\|^2}\right) > \frac{1}{\|h_{M-1}\|^2}. \tag{2.9}$$

Then it can be shown that the optimizing g_k obey:

$$g_k = \begin{cases} \frac{h_k^*}{\|h_k\|}\sqrt{\frac{1}{M}\left(n + \sum_{l=0}^{M-1}\frac{1}{\|h_l\|^2}\right) - \frac{1}{\|h_k\|^2}} & 0 \leq k \leq M \\ 0 & \text{else} \end{cases}. \tag{2.10}$$

This is in the vein of most classical water filling solutions. Larger n and/or large SNRs means fewer subchannels are silent.

3. THE MAIN RESULT

The summation in each logarithm term in (2.4) is over the channel index k, while the constraints in (2.5) are summations in the transmitter index i. Thus one cannot conclude that the matched filtering condition (2.8) results in optimality for Problem 2.1. Instead, as (2.5) only involves the magnitudes of the g_{ik}, maximization requires that for some real α_{ik}

$$g_{ik} = \alpha_{ik}^2 h_{ik}^*. \tag{3.11}$$

Then Problem 2.1 becomes:

Problem 3.1 *Given complex scalar h_{ik}, find scalar α_{ik} to maximize*

$$\sum_{k=0}^{N-1}\ln\left(1 + \left(\sum_{i=1}^{n}\alpha_{ik}^2|h_{ik}|^2\right)^2\right) \tag{3.12}$$

subject to:

$$\sum_{k=0}^{N-1}\alpha_{ik}^4|h_{ik}|^2 = 1, \ \forall\, i \in \{1,\cdots,n\} \tag{3.13}$$

To avoid trivialities we make the following assumption.

Assumption 3.1

$$h_{ik} \neq 0, \ \forall\, i \in \{1,\cdots,n\}, k \in \{0,\cdots,N-1\} \tag{3.14}$$

Observe that Problem 2.1 has been reduced to an nN-variable, real, constrained optimization. The fact that (2.8) is not optimizing may seem to preclude the possibility of reduction to the N-variable optimization that Problem 2.2 reduces to. Yet the sequel shows that not only does Problem 3.1 and hence Problem 2.1 reduce to an N-variable optimization problem but it does so to an *unconstrained one*.

There are two key mathematical features of Problem 3.1 that set the stage for the results here. First, each summand in the utility function (3.12) is a *per subchannel* summand, while each constraint in (3.13) is a *per user* constraint. Thus, there are as many summands in the utility function, as there are subchannels, just as there are as many constraints as there are users.

The second feature is that while the utility function is quadratic in the α_{ik}, the constraints exhibit a quartic dependence. The first consequence of these features is Theorem 3.1 which asserts that, just as in Problem 2.2, should a transmitter be silent on a given subchannel, then *all* transmitters must be silent on this same subchannel. Due to space constraints we provide only a proof outline.

Theorem 3.1 *Consider Problem 3.1, with Assumption 3.1 in force. Suppose for some $i \in \{1,\cdots,n\}$ and $k \in \{0,\cdots,N-1\}$, the*

optimizing $\alpha_{ik} = 0$. Then for this k and all $l \in \{1,\cdots,n\}$, the optimizing $\alpha_{lk} = 0$.

Proof: Suppose $\alpha_{10} = 0$, but $\alpha_{20} \neq 0$. As each transmitter must transmit on at least one subchannel assume without loss of generality that $\alpha_{11} \neq 0$. The resulting utility function is for some $K \geq 1$, $\rho_1 > 0$ and $\rho_2 \geq 0$

$$J_0 = \ln\left(1 + \rho_1^2\right) + \ln\left(1 + \left(\rho_2 + h_{11}^2\alpha_{11}^2\right)^2\right) + \ln K. \tag{3.15}$$

Now consider the value of the utility function obtained by choosing $\alpha_{10} = \epsilon$, keeping all other α_{ik} unchanged excepting α_{11}, which is adjusted to satisfy (3.13) for $i = 1$. Under this new allocation the derivative of the resulting utility function, $J_1(\epsilon)$ with respect to ϵ, takes the form of

$$J_1'(\epsilon) = \epsilon f(\epsilon) - \epsilon^3 g(\epsilon)$$

where both $f(\epsilon)$ and $g(\epsilon)$ approach positive constants as ϵ tends to zero. Thus for some ϵ^* and for all $0 < \epsilon < \epsilon^*$ this derivative is positive, i.e. increasing α_{10} to a sufficiently small positive value increases $J_1(\epsilon)$. ∎

We now provide the separation property and only a proof outline.

Theorem 3.2 *Consider Problem 3.1, with Assumption 3.1 in force. Then there exist scalars a_i, $i \in \{1,\cdots,n\}$ and b_k, $k \in \{0,\cdots,N-1\}$, such that at the optimum:*

$$\alpha_{ik}^2 = a_i^2 b_k^2. \tag{3.16}$$

Further at least one $b_k \neq 0$. The scalars a_i obey

$$a_i^2 = \frac{1}{\sqrt{\sum_{k=0}^{N-1} b_k^4|h_{ik}|^2}}. \tag{3.17}$$

Finally the b_k maximize

$$\sum_{k=0}^{N-1}\ln\left(1 + b_k^4\left(\sum_{i=1}^{n}\frac{|h_{ik}|^2}{\sqrt{\sum_{l=0}^{N-1} b_l^4|h_{il}|^2}}\right)^2\right). \tag{3.18}$$

Proof: Consider the pertinent Lagrangian

$$\sum_{k=0}^{N-1}\ln\left(1 + \left(\sum_{i=1}^{n}\alpha_{ik}^2|h_{ik}|^2\right)^2\right) - \sum_{i=1}^{n}\lambda_i\left(\sum_{k=0}^{N-1}\alpha_{ik}^4|h_{ik}|^2 - 1\right) \tag{3.19}$$

Under Assumption 3.1, if $\alpha_{pq} \neq 0$, KTT conditions yield

$$\alpha_{pq}^2 = \frac{1}{\lambda_p}\frac{\sum_{i=1}^{n}\alpha_{iq}^2|h_{iq}|^2}{1 + \left(\sum_{i=1}^{n}\alpha_{iq}^2|h_{iq}|^2\right)^2}.$$

Thus α_{pq}^2 is the product of a term indexed just by p and a term indexed just by q. Thus indeed (3.16) holds when $\alpha_{pq} \neq 0$. In view of Theorem 3.1 it also holds with $b_q = 0$ when $\alpha_{pq} = 0$. Substituing into (3.13) yields (3.17). Substituting into (3.12) yields the utility function in (3.18). ∎

Theorem 3.1 is a special case of Theorem 3.2, though it is needed to prove Theorem 3.2. The scaling of a vector of b_k doesn't change the utility function in (3.18).

Further (3.16) is the separation property, and the solution of Problem 2.1, under (3.11) is accomplished by the N-variable, real,

unconstrained optimization of (3.18). The solution is in fact modular: Find b_k to maximize the function in (3.18). Find a_i using (3.17). Then with α_{ik} as in (3.16), (3.11) provides the optimizing g_{ik}.

4. SIMULATIONS

Simulations are for two scenarios corresponding to low and high SNR. Three power allocation constraints are considered: (a) constraint on average total power summed across transmitters ('Total Power Constraint'), (b) separate constraint on the average power of each transmitter ('Per Transmitter Constraint') and (c) equal power allocation for each channel on each transmitter ('Equal Power'). For the "low SNR" case we consider $N = 4$ subchannels with the channel gains for each user on each channel chosen iid $\sim CN(0, \sigma_l^2)$, where $\sigma_l^2 = -20, -13, -10$ and -7 dB for the first, second, third and fourth channel respectively. Receiver noise level is 0 dB.
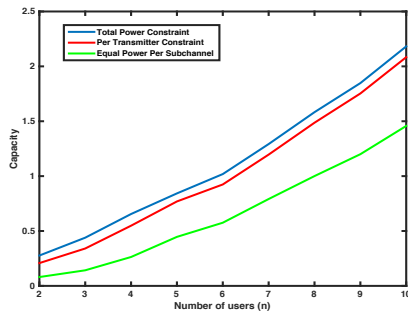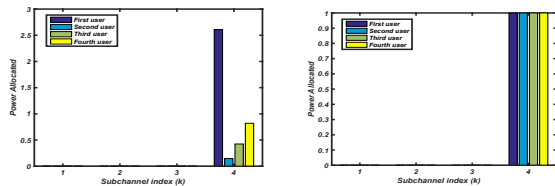

Fig. 1: Capacity for low SNR as a function of number of users with four subchannels.

Figure 1 depicts capacity as a function of n, the number of users, and shows that while the performance loss from the centralized to the decentralized case is modest, the disparity of both from the equal allocation case is very substantial and grows with n. It also depicts the monotonic increase of capacity with n. This is expected as the average received SNR scales as n^2.

For small n, and low SNRs (2.10) suggests that in the centralized case users should be silent on weaker channels. This is confirmed by Fig. 2b which shows that with four users all power is allocated by each to the strongest channel. Figure 2 shows that this is true even in the decentralized case, confirming a consequence of the separation property that should a user be silent on a particular subchannel then all users are silent on this subchannel.

Figure 3 considers the high SNR case with four subchannels, chosen for each user as iid $\sim CN(0, \sigma_h^2)$, with $\sigma_h^2 = 40, 47, 50$ and

(a) Total Power Constraint.   (b) Per Transmitter Constraint.

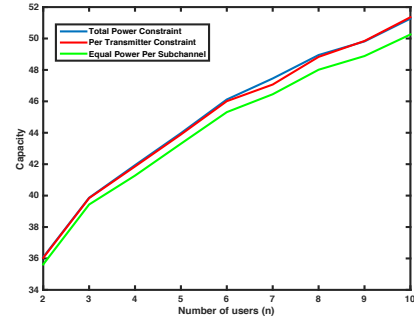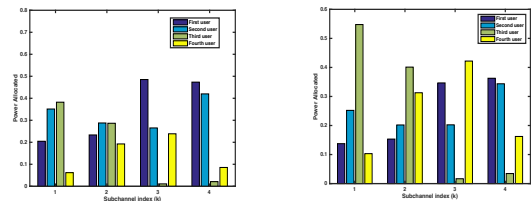**Fig. 2**: Power allocated to channels by each user at low SNR.

Fig. 3: Capacity for high SNR as a function of number of users with four subchannels.

(a) Total Power Constraint.   (b) Per Transmitter Constrain.

Fig. 4: Power allocated to channels on each user at high SNR.

53 dB for the first, second, third and fourth channel respectively. We see from Fig. 3 that the capacity increases with increasing n in a concave (logarithmic) manner, as we expect. The decentralized performance is virtually indistinct from the centralized case. Disparity with the equal allocation case still persists but is less pronounced. As shown in figures 4a and 4b, at this high SNR case no channel is silent. This is evident from (2.10) for the centralized case. Evidently the trends seen for the centralized case also manifest the decentralized case.

5. CONCLUSION

Motivated by the goal of obtaining theoretical benchmarks for distributed beamforming over frequency selective channels we have characterized conditions under which capacity is maximized in an OFDM framework subject to the constraint that all transmitters transmit with equal power. The key analytical take away is that the optimal precoders obey an attractive separation property that converts this nN variable, complex constrained maximization problem to a N-variable real, unconstrained optimization problem. We compare performance with a hypothetical centralized problem where capacity is maximized subject to a constraint on the total power transmitted across all users. The decentralized problem's performance tracks and is close to the upper bound provided by the centralized problem. At low SNRs both significantly outperform the case where each user allocates equal power on each subchannel. The performance disparity at high SNRs is smaller but still discernible.

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